

Reliability, Maintainability, and Availability
for Engineers

Text Book



Defense Acquisition University
Mid-West Region

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Reliability, Maintainability, and Availability For Engineers

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Preface

This textbook is intended to supplement the Reliability, Maintainability, and Availability for Engineers course class presentation. Professor Virgil Rehg originally developed this material for the Air Force Institute of Technology School of Systems and Logistics course, QMT 372 Reliability.

Any errors can be attributed to the undersigned. Comments for improvement would be appreciated.

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Reliability, Maintainability, and Availability (RMA) For Engineers: Description

This course is a tailored and updated sub-set of the (former) AFIT School of Systems and Logistics QMT 372 Reliability course. It is tailored to comply with NAVSEA Port Hueneme Division request for an off-the-shelf RMA for Engineers course.

The course also:

1. Provides the participant with an understanding of the principles and assumptions of reliability, maintainability, and availability (RMA). Emphasizes a study of the time-to-failure distributions used in reliability: normal, exponential and Weibull.
2. Covers reliability allocation and prediction techniques for equipment and systems; reliability growth and reliability qualification testing, including the development of O.C. (operating characteristic) curves and the use of relevant Military Handbooks for reliability testing.
3. Includes the binomial and Poisson distribution applications to reliability.

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Chapter 1

INTRODUCTION TO RELIABILITY

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1.1. Why Study Reliability?

Anyone who has owned a car, a TV set, a washing machine, or any kind of modern equipment knows why reliability is important.

Just imagine you are riding in a car on a trip. One of the last things you want to have happen is for the engine to stop running, the headlights fail if it is at night, the radiator to overheat, a tire to blowout, or anything else that could prevent you from having a successful trip. To avoid situations such as these, design engineers and reliability engineers work together early in the design stage of the equipment to design and develop:

- a cooling system that maintains a constant temperature in all environments
- tires that do not fail and provide a long life,
- an engine that will last a long time and have a low chance of failure in all environments;
- a lighting system that is not affected by moisture, heat, cold, mildew, vibration, age, shock,

so that cooling system, the tires, the engine, and the lighting system work, and continue to work, in all environments that could be experienced during the trip. Similar efforts are used in the design and development of other systems in the vehicle, e.g., the hydraulic system, steering system, power system, exhaust system, starting system, and so on.

1.2. Definitions

The examples in the preceding paragraph should illustrate why reliability is important and some of the potential problems that must be overcome in the design. The reliability engineer's job is to help an organization produce a product that meets the

customers' requirements. A summary of what reliability entails can be seen from the formal definition of reliability:

RELIABILITY: Reliability can be described as a discipline related to the **design, development, test, and manufacture** of an item, so that it successfully performs a certain **task**, under specified **conditions**, for a certain length of **time** or number of cycles with a specified **probability**.

The words task, conditions, time, and probability are key words in the definition.

The task is the job the equipment is designed to perform. It could be transporting people, keeping food cold, removing dirt from clothes, etc., depending on the equipment being designed.

The conditions are the environments that the equipment will experience at some time during its life which includes performing its task. Conditions include vibration, temperature, humidity, altitude, shock, sand and dust, etc.

Time is the length of the mission. This can be in seconds, hours, days, years, etc. Time may be different for different parts of the system. But not every item of the system is measured in time, some are measured in cycles. A switch, for example, may be required to turn off and on several times during a mission, where each on-off sequence is called a cycle.

Probability is a numerical value that expresses the proportion, or per cent of the time the equipment will perform its mission successfully. It is called the probability of success, or the reliability for a mission of "t" hours, $R(t)$. This number should be high, that is, close to 1.0, or 100%. This probability is illustrated graphically on the probability density function (p.d.f.) as the area under the curve and to the right of the mission duration. Figure 1.1 shows a special case - the exponential density function.

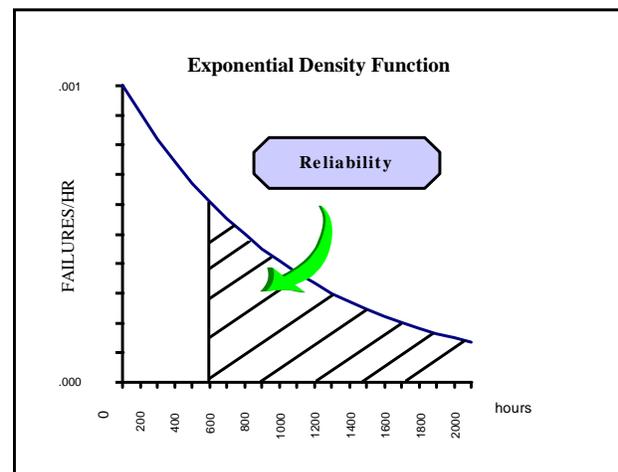


Figure 1.1 Graphical Illustration of Reliability

The words design, develop, test and manufacture, may lead you to believe that reliability applies only to new equipment. This is not the case, reliability techniques can be applied to equipment that is being repaired or modified, and can also be applied to services that are performed.

It is just as important that equipment being repaired be brought to the same level of reliability as it was when it was new, as it is to get reliability in new equipment. This involves

maintainability.

1.3. Maintainability

Maintainability must be considered in conjunction with reliability because preventive and/or corrective maintenance is performed on most equipment throughout the equipment's life. And you want the equipment to be at least as reliable after the maintenance as it was before the maintenance. For example, the oil in a car is changed to extend its life. The expectations related to this type of maintenance are that: the correct oil is used, the right amount is put in the car, the oil plug is screwed in correctly and tight, a new oil filter is installed, and nothing else is damaged in the performance of this maintenance. You would also expect that this job will be completed in a certain length of time. The formal definition for maintainability is as follows:

MAINTAINABILITY: The **probability** that an equipment will be **retained** in, or **restored** to, a specified **condition** in a given period of **time**, when maintenance is performed in accordance with prescribed **procedures** and **resources**.

The key words are: probability, retained/restored, condition, time, procedures, and resources.

Time in this definition refers to time it takes to perform the maintenance and is a function of the equipment design. This means that the length of time that a maintenance task is going to take must be considered when equipment is designed. Usually two times are identified during the design stage: the average time to perform maintenance; and the maximum time it will take to repair almost anything on the equipment. A high probability, usually around 95%, is attached to this second time. For example, the user of the equipment may have a requirement that the equipment be designed so that the average repair time is two hours, and that 95% of all possible repair actions can be completed within five hours.

Time in the maintainability definition includes the time it takes: getting access to specific parts of the equipment, trouble shooting (diagnosing), making a repair and/or a replacement, calibrating, testing, and closing up the access panels.

The word probability is used to express the proportion (or per cent of the time) that a maintenance task is performed in a specified time. For example, suppose you are a mechanic in the motor pool.

Your supervisor brings you a task and asks what is the chance that it will be finished within an hour. If you say 80%, you are saying that there is a good chance of doing the work in an hour; but if you said 99%, there is a very high chance that you will be finished in an hour. For any task, and for any repair time, there is a probability the job will be done in that time. Figure 1.2 is a graphical illustration of where the probability occurs on the probability density function for maintainability.

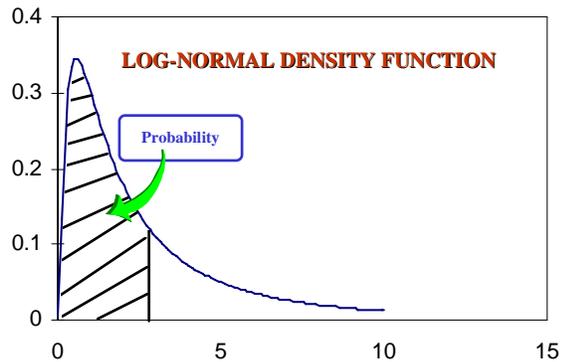


Figure 1.2 Graphical Illustration of Maintainability

The probability related to any repair action depends on several factors, including: the way the equipment is designed, the tools available, the skill of the person doing the repair, the environment, technical manuals, the ease of doing trouble shooting and getting access to the critical area, and the motivation of the person doing the repair.

Retained in, or restored to are words used in reference to preventive or corrective maintenance. The expectation is that when either type of maintenance is performed, the equipment is brought back to its original condition.

Condition can refer to the original design level. However, for some systems it is expected that there will be some degradation of the system. A car, for example, is not expected to have the same pep at 100,000 miles that it had at 5,000 miles. Therefore, condition can be a relative term depending on the system.

Procedures are the instructions and methods used to perform maintenance. This includes manuals that describe how to perform maintenance, and the standard procedure for the maintenance. Procedures can be a function of the environment. For example, certain maintenance tasks can be performed outside in warm weather but must be done inside when the weather is cold or if it is raining. Hence there is another step in the process that could affect the maintenance time.

Resources include the skills of the individuals performing maintenance, and the tools used by these individuals. Skill

level and type of tool affect the maintenance time.

Maintainability is an important discipline for both reliability engineers and design engineers. However, this text is devoted primarily to reliability models and methods. It is important that maintainability is included at this time because it is important that reliability engineers be aware of the affect that the design has on the maintenance process and system availability.

1.4 Bathtub Curve

The bathtub curve gets this name from its shape. It is a graph of the failure rate over the life of a system. In the late fifties, as part of the AGREE (Advisory Group for the Reliability of Electronic Equipment) study, it was discovered that the failure rate pattern over the life of electronic equipment could be explained by the bathtub curve. The horizontal scale of the bathtub curve is time, and the vertical scale is failure rate. The bathtub curve can be divided into three sections, the infant mortality stage, the useful life stage, and the wear out stage. Mechanical systems usually do not have a constant failure rate; hence, their life curves have a different shape.

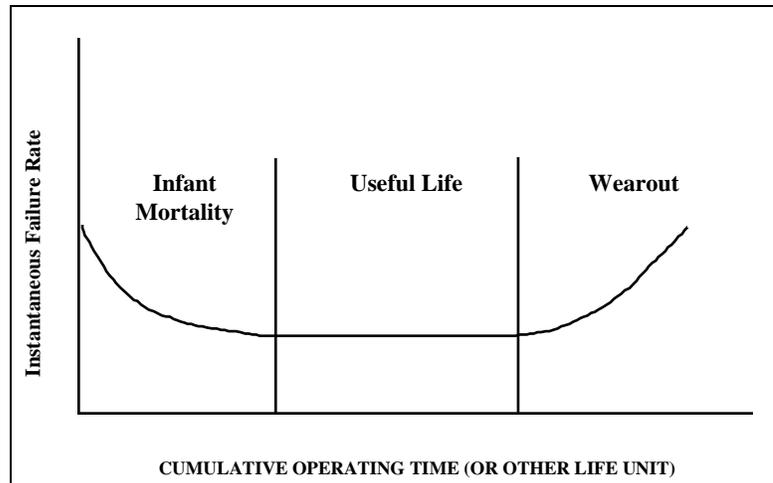


Figure 1.3 Bathtub Curve

INFANT MORTALITY: The infant mortality stage is so called because the failure rate is high just as it is at birth for human beings. For equipment, the failure rate is high because of errors made in the manufacturing process. If the manufacturing process could build systems without making mistakes, there would be no infant mortality stage; the useful life part of the curve would start at time zero.

The types of mistakes made during the manufacturing stage include the use of wrong parts during assembly, using parts that do not meet requirements, using unskilled and/or untrained operators to manufacture parts or build assemblies, purchasing low quality raw materials, using inadequate procedures, using the wrong tools, or making any kind of a mistake that could impact the manufacturing and assembly process that could cause the system to fail in use. See Figure 1.3.

To eliminate the infant mortality stage of the bathtub curve, errors made in the manufacturing process must be eliminated. However, the errors are a symptom of a deeper

problem that needs attention, the lack of quality. To improve quality requires a sincere effort to improve processes throughout the organization and a change in the cultural climate which must start at the top.

This is not a simple task; it means that management must be involved in creating a **vision** of where the organization is headed and how it is going to operate in the future. This vision must be **communicated** throughout the organization and be **shared** by the workers. There must be a **plan** that gets the organization working toward the vision, and workers must be **empowered** to make changes that improve the quality of the processes.

For statistical purposes, the infant mortality stage of the bathtub curve can be described by the Weibull distribution.

USEFUL LIFE: The middle part of the bathtub curve in Figure 1.3 is the useful life stage. When the useful life is a horizontal line, the failure rate for the system is constant. This line represents an average, implying that there are times when the failure rate is higher than the average and times when the failure rate is lower than the average. A corollary would be driving one-hundred miles in two hours, which means you averaged fifty miles per hour. But that does not mean that at every instant of time you were going fifty miles an hour. Sometimes you may have been going sixty, and at other times forty, but on the average you were going fifty.

The instantaneous failure rate is the failure rate at a specific point in time, and it is also called the hazard rate, $h(t)$.

The height of the useful life portion of the bathtub curve corresponds to the failure rate of the system. Ideally, this is the failure rate designed into the system.

The time at which the useful life ends is the system's life; it is the time when wear-out begins. There are mathematical models that can be used to predict when the useful life may end, that is, when wear-out begins. However, these models should be improved once you have actual failure times.

During the useful life stage of systems with a **constant failure rate**, the probability of failing is the same at any point along the useful life line. Hence, an item that just entered the useful life stage and an item that is almost at wear-out would have the same failure rate.

The useful life stage of the bathtub curve can be described by the exponential distribution if the failure rate is constant, and by the Weibull distribution if the failure rate is not constant. However, it should be noted that the Weibull can even be used when the failure rate is constant.

WEAR OUT: The wear out stage of the bathtub curve has an increasing failure rate and this means that the chance of failure

increases.

The wear-out phase can be delayed by a good preventative maintenance plan. By preventing failures, the wear-out phase can be pushed further into the future.

The wear out stage can be defined by the normal distribution or the Weibull distribution.

1.5. System Requirements

To design, develop and produce a reliable product involves a strategic plan. It requires:

1. Complete and clearly defined requirements from the customer.
2. A practical design based on customer's needs, in which reliability prediction models are used to make trade-offs between various design choices.
3. A design that can be easily maintained at a low cost.
4. Accurate historical data.
5. A reliability development and growth testing program that identifies design weaknesses.
6. A reliability testing program to prove the adequacy of the design under actual conditions, or a simulated environment that duplicates actual conditions.
7. A manufacturing process built on the concept of continuous improvement; and the use of statistical methods as a means of reducing process variation.
8. A culture in the organization that creates an environment conducive to communication, team work, and high quality levels in all processes.
9. A corrective action system capable of identifying root causes of problems quickly and effectively.
10. A management team that empowers the work force, is supportive and willing to listen to new ideas, and considers the employees as its most important asset.
11. A management philosophy in the organization committed to continuous improvement.
12. An organization driven by a vision that benefits the workers, customers, suppliers, and society.

There may be organizations that produce reliable equipment without using a management style that practices these twelve points. But these organizations succeed in spite of themselves

and could be even more effective if they adopted some of the concepts in the twelve points listed above. In general, an organization that follows these points should have a better chance of producing reliable products, have less turnover, communicate more effectively, be a healthier organization, and have a better chance for long term survival.

1.6. Summary

The purpose of this section is to give the reader an overview of what is involved in producing reliable equipment. Strong efforts must be made during the design stage and in establishing a high quality manufacturing capability. In the past, most of the effort has been to concentrate on design and give less attention to quality. However, from various studies it has been shown that more failures in operation are caused by the manufacturing process than by the design process.

The remaining sections of the text are statistical in nature. However, the approach is practical rather than theoretical. Math models will not be developed, instead, models that have been developed will be used to show how they apply to design, development, testing, and the manufacturing of reliable products.

1.7. Appendix

THE BIG PICTURE

The purpose of this appendix is to discuss the "big picture," to show how the reliability and quality models and techniques used in this course are related. The big picture is an overview of the course content in terms of the models and techniques taught during the course and their application relative to the acquisition process.

COURSE CONTENT

The course content can be divided into the following categories:

- Statistical models used in R/M
- Reliability and Maintainability models and techniques
- Planning for Quality
- Management of the R/M process
- Maintenance Concept

STATISTICAL MODELS USED IN R&M:	
MODEL	APPLICATION
Probability Rules	Prediction, testing
Discrete Models:	
Binomial	Prediction, testing, confidence statements, sampling, control charts, computing reliability
Poisson	Testing, OC curves (Operating Characteristic), setting inventory levels, computing reliability, control charts
Hypergeometric	Sampling
Continuous Models:	
Exponential	Allocation, prediction, computing reliability, confidence statements
Weibull	Analyze data, confidence statements, prediction, testing (not part of course)
Normal	Sampling, control charts
Log-Normal	Maintenance prediction (not part of this course)

RELIABILITY & MAINTAINABILITY MODELS AND TECHNIQUES:	
Allocation (apportionment) - Equal apportionment model	To distribute a system requirement throughout the system (allocations starts with the system requirement and you apportion down, level by level, until you reach the part level in a system)
Prediction	To estimate the reliability of a system (prediction starts at the lowest level of the system for which you have data and you compute probabilities until you reach the system level)
Reliability Development Growth Testing (RDGT) via Test, Analyze and Fix (TAAF) - Duane model	To raise the Mean Time Between Failures (MTBF) up to an acceptable level
Reliability Qualification Test (RQT)	To demonstrate, during the Development Phase, that the system reliability meets the system requirement
Production Reliability Acceptance Test (PRAT)	To demonstrate that production units meet the system reliability requirement
Environmental Stress Screening (ESS)	To find units that have not been manufactured correctly so they may be fixed before they are tested and/or sent to the customer

PLANNING FOR QUALITY:	
In Design:	
FMEA (Failure Mode and Effects Analysis)	To prevent failures
FMECA (Failure Mode, Effects and Criticality Analysis)	
Design Review	To prevent problems in design, production, testing, etc.
Human Factors Engineering	To ensure that the equipment can be used by the operator successfully
Other techniques/models that have application but are not part of this course include: Quality Function Deployment (QFD) System Design Parameter Design Tolerance Design Concurrent Engineering	
In Manufacturing:	
Statistical Process Control - X-bar and R charts - fraction defective charts - defects charts	To help the production line and assembly lines meet the print specifications and operational requirements
Continuous improvement process	To achieve a reduction in process variation in all processes
Defect prevention	To prevent process errors before they occur
Corrective action system	To assure that permanent fixes are made when errors are found in the system
Capability Studies	To determine the capability of manufacturing (and other) processes
Variation Reduction	To reduce variation in all processes
Empowerment	To authorize process action teams to make improvements in their processes
Other Quality Functions that are not part of this course:	
Incoming products and services	To assure the quality of the incoming products/services
Quality Manual	To describe various procedures related to improving and maintaining quality in products and services
Calibration System	To ensure that all gages are accurate

MANAGEMENT OF THE R&M PROCESS:	
Reliability Program Plan Tasks	To ensure that the activities required to achieve reliability are being done accurately and completely
Data Item Descriptions	To ensure that the procuring activities buys enough information so they can properly manage the process

MAINTENANCE CONCEPT:
The selection of a maintenance plan involves the selection of the level to which maintenance is to be performed, the location of the replace and repair actions, and whether the unit removed will be replaced discarded. These decisions determine the MTTR (Mean Time to Repair and the cost of the maintenance plan.

Chapter 2

INTRODUCTION TO PROBABILITY

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Symbols, Terms and Definitions:

Point Estimate - estimate of a population parameter

Prediction - forecast of what is expected

Block Diagram - graphical illustration of a reliability model

Binomial - discrete distribution

Exponential - continuous distribution

Density function - mathematical expression of a data pattern

Continuous model - category of distributions

Parameters - population measures

Poisson - discrete distribution

Normal - continuous distribution

Weibull - continuous distribution

Factorials - a method of counting certain events

Base e - 2.71828+

In the definition for reliability it is stated that reliability is a probability. Computing this probability for a system depends on the history that is available for the system and the purpose for computing the probability.

2.1. Probability based on past history

If the number of successes and the number of trials for a system are available, an estimate of the system's reliability is calculated as follows:

$$\text{Reliability} = \frac{\text{Number of successes}}{\text{Number of trials}} \quad (2.1)$$

Successes could be based on actual use, or based on controlled tests but they should not be mixed. In either case, it is important that the definition of what is meant by a "success" is clear, and that the definition is consistently applied by all those involved in determining the number of successes. It is also important that the definition of what is meant by a system is clear and is applied in a consistent manner.

The "probability of success" and the "probability of no failures" are other terms sometimes used in place of "reliability." In this text these terms will be used interchangeably.

POINT ESTIMATE: It is important to recognize that the result of equation 2.1 is an estimate of the system's true (or population) reliability, and as such is subject to sampling error. The only way to avoid sampling error is to test the entire population of systems until they all fail and then compute the system reliability. Since this usually is neither feasible nor practical, point estimates of the true reliability are commonly used.

This estimate is called a point estimate because it is a single point on the reliability scale that goes from 0 to 1.0. We know that it will be inaccurate but we hope it will be close to the true reliability. Unfortunately, we do not know how close the point estimate is to the true reliability. In the chapters that follow confidence intervals will be discussed, and they help us avoid some of the deficiencies of point estimates.

2.2 New Systems

A new system is one for which there is no history on the system as such. This situation arises early in the life cycle of a new system, when it is in the design or development stage.

PREDICTION: The process for estimating the reliability of a new system is called prediction. In the prediction process, we start at a level of system for which historical data is available and build the system piece by piece, and level by level, using the laws of probability and probability density functions, until we reach the system level.

Hence, even for new systems historical data must be available before the prediction can be made. Again, the historical data can be based on actual operations or controlled tests.

Note: The levels of a system include but are not limited to: the system level, the subsystem level, the assembly level, the subassembly level, and the parts level.

LAWS OF PROBABILITY: At each level of the system there are certain things that must occur for a mission to be successful. To compute the reliability at each level, we begin by building block diagrams and then apply the laws of probability to the block diagrams.

The block diagram is used for reliability purposes and is not the same as the functional design configuration. For example, if you were driving a car at night, the headlights and the engine would be on at the same time. For functional purposes they are in parallel because at night they operate at the same time. But for reliability purposes they would be in series because both of them must operate for the car to have a successful mission. Hence, the reliability block diagram would show these two blocks as follows:

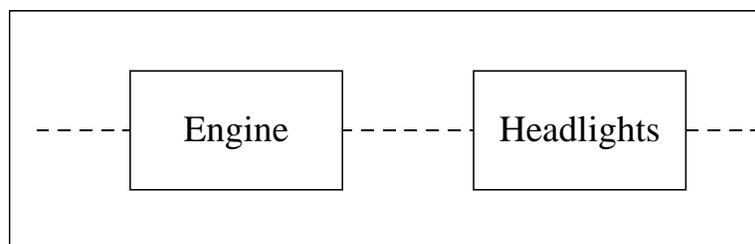


Figure 2.1 Reliability Block Diagram for the Engine and Headlights for a Car in darkness.

The probability density function (p.d.f.) is used to compute the reliability for the elements of a system. There are many different density functions, and the one that is used depends on the nature of the element. For example, an element used to turn something on or off, such as a switch, has a failure pattern that is usually described by a binomial density function; an electronic component has a failure pattern that is described by an exponential density function.

PREDICTION PROCESS: To make a reliability prediction for a specified element of a reliability block diagram at a specified level, the following steps are used:

1. Identify the element in the system that is under study.
2. Select the density function that fits that element; this will be an equation.
3. Identify the parameters of the equation.
Note: Parameters are the unknown in the equation that you need to know to compute probability.
4. Collect the historical data for the element. These will be statistics and are used to estimate the parameters needed in the equation.
5. Insert the historical data into the equation and compute the probability.

From the preceding discussion, it should be evident that we need to know several things to estimate reliability for a new system, including:

1. How to select a probability density function.
2. How to use a probability density function to compute the reliability for an element.
3. The reliability block diagram.
4. Laws of probability.

In the chapters that follow the laws of probability, the probability density functions, and reliability block diagrams are presented. In a later chapter there is more discussion of the prediction process.

2.3. Laws of Probability

The laws, or rules of probability that are presented in this text can be summarized as follows:

1. Basic rule.
2. Product rule.
3. Addition rule.
4. Rule of complements.

2.4. Probability Density Functions

Density functions are either discrete or continuous. Discrete models are used with counted observations, for example, the number of failures, the number of defects in a system, the number of defectives in a sample, etc. In each case the observation is a whole number, hence the term "counted observations."

Continuous models are used with measurements. A measurement always involves some type of measuring device, for example, a

clock, a volt meter, a water meter, a ruler, a scale, etc. These observations are not whole numbers unless the data collector elects to round off an observation to some whole number, like three pounds instead of 3.1 pounds. The primary difference between a measurement and a counted observation is that a measurement can take on any value, but a counted observation is always a whole number. Measurements, however, are limited by the device used to perform the measurement, and for that reason the exact measurement is never known.

2.5. Choosing a Model

The choice of a model, that is choosing the most appropriate model (either continuous or discrete), depends on the following:

1. Type of data: discrete or continuous.
2. Historical applications: models that have been used in the past for similar data.
3. For continuous data, the pattern or shape: A tally sheet or histogram of continuous data is helpful if the sample size is large.
4. For discrete data the following factors are important:
 - a. Lot size, finite or infinite.
 - b. Whether the probability of an event is the same from trial to trial?
 - c. Number of possible ways an event can occur.
 - d. Whether the probability of the event is small?
5. Properties of the density function can also be used to help select the correct model.

One of the best indicators of the correct model is past history. If it is known that a certain model usually works in a certain situation, then it probably is the correct choice. This can be verified if data is available to construct a histogram, or perform a goodness-of-fit test. If past history or data is not available, typical applications or properties of various models can be examined.

2.6. Probability Calculations for Discrete Density Functions

Several situations are possible in computing the probability for discrete functions, and they include the probability of:

1. An exact number of occurrences, e.g. $r = 2$, where "r" is the number of occurrences.

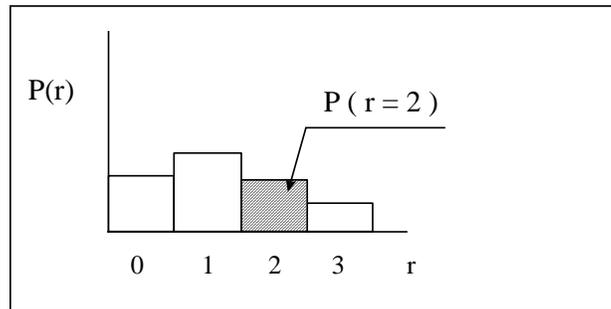


Figure 2.2 An Exact Number of Occurrences.

2. "r" or more occurrences; e.g. $r \geq 2$, which means "r" is equal to or greater than 2; in other words "r" is at least 2.

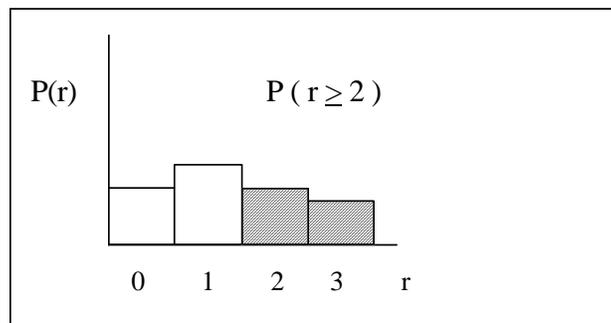


Figure 2.3 At Least Two Occurrences.

3. More than "r" occurrences; e.g. $r > 2$, which means "r" is more than 2; in other words, 3 or more, or at least 3.

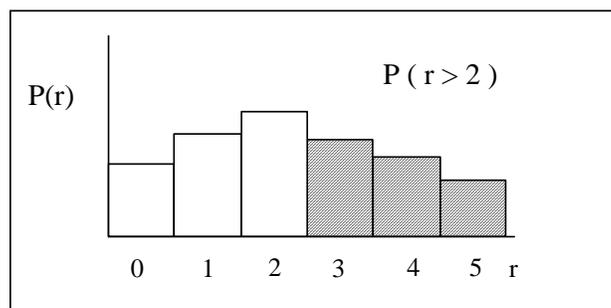


Figure 2.4 More Than Two Occurrences.

4. "r" or less occurrences; e.g. $r \leq 2$, which means "r" is 2 or less; in other words at most 2; or "r" is equal to or less than 2.

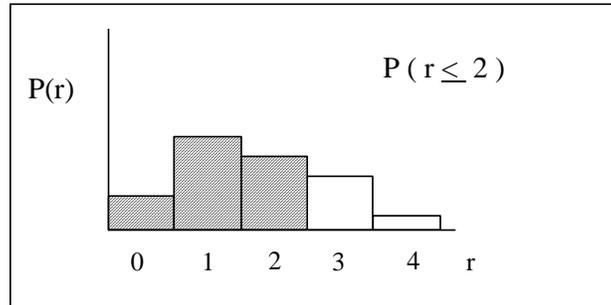


Figure 2.5 At Most Two Occurrences.

5. less than "r" occurrences; e.g. $r < 2$, means "r" is less than 2; in other words at most 1; or "r" is 1 or less.

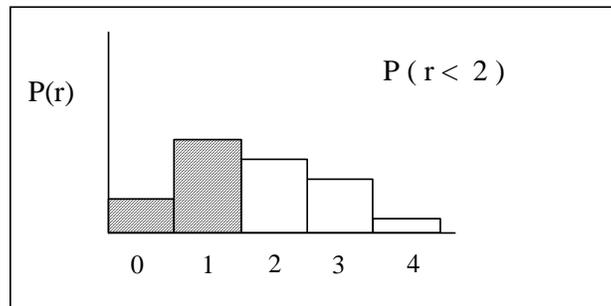


Figure 2.6 Fewer Than Two Occurrences.

There are other ways to state these situations but the descriptions given are commonly used. To see what these situations look like, the density functions for these five cases are illustrated above:

2.7. Parameters

Probability density functions are defined by parameters. A parameter is a descriptive measure of the function. To compute probability you must know what the parameter is, or have an estimate of the parameter.

The number of parameters varies for each density function. The density functions to be studied in this course have the following parameters:

Discrete Density Functions:	Parameter(s):
Binomial	n (sample size) p (fraction defective)
Poisson	m (expected number of occurrences of an event)
Continuous Density Functions:	Parameters
Normal	(mean) (standard deviation)
Exponential	(mean time between failure)
Weibull	(shape parameter) (scale parameter) (location parameter)

From this list it is evident that some density functions have one parameter, some have two, and one has three parameters. The important point is that you must have a value to substitute for the parameter in the density function. That is the only way you can compute probability.

2.8. Factorials

Factorials are used in some density functions. The notation for a factorial is the exclamation point (!) that follows a term in the model, e.g. "n!"

The factorial symbol simply means that you must do successive multiplication beginning with the number the factorial symbol (!) follows until you get to "1". For example,

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \dots \times 1 \quad (2.2)$$

$$\text{If } n = 3, \text{ then } 3! = 3 \times 2 \times 1 = 6$$

2.9. Base e

In some of the density functions the base of the natural logarithm system, e, is used. The value of e is 2.71828+. In making calculations using "e," you may either plug in the value 2.71828, or you use the "e" key on a calculator, if you have a scientific calculator. If you do not have an "e" key, look for a key labeled "ln" or natural logarithm, use the inverse or second function key. The inverse of "e" on the calculator is the natural logarithm, or "ln."

2.10. Summary

This chapter is an introduction to the chapters where probability is calculated. The basic concepts presented here should make the study of probability easier because you will already know about some probability and density functions.

This information is presented at this time so that you will know why we spend time going through the various methods of computing probability. In a sense, the big picture has been presented, now our task is to look at some of the details.

Another reason for this chapter is that the density functions to be examined have been listed so you know what is coming, and you know something about a parameter.

A final point to remember is that each time we are looking for a probability, we are actually looking for an area under the density curve. The only difference in the calculation is in the density function that is used. In making the calculations we will use tables as much as possible to save time, and perhaps avoid making an error in the calculation.

2.11. Equations

$$\text{Reliability} = \frac{\text{Number of successes}}{\text{Number of trials}} \quad (2.1)$$

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \dots \times 1 \quad (2.2)$$

Chapter 3

PROBABILITY

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Symbols, Terms and Definitions

- m = the ways an event can occur, or has already occurred,
 n = the ways an event can fail to occur, or has already done so,
 $n!$ = n factorial, and it tells you to multiply all the numbers from " n " down to " 1 ," for example,
 $3! = 3 \times 2 \times 1 = 6$
 $P(A)$ = Probability that event A will occur,
 $P(B|A)$ = Probability that " B " will occur on the assumption that " A " has already occurred,
 $P(S)$ = Probability of Success
 $P(F)$ = Probability of Failure
 N = population size,
 D = number of defectives in the population,
 x = number of defectives in the sample.

3.1. Introduction

Since one of the key words in the definition for Reliability is **probability**, it seems appropriate to begin with a discussion of some of the basic concepts of probability.

The probability scale goes from zero to one. A probability of zero means the event cannot occur; and a probability of one means that the event will always occur.

Everyone uses probability every day of their lives in the decisions they make. Choosing routes for getting to work, deciding which stores to patronize are often based on probability. For example, you choose a certain route to work because it gives you the best chance of: getting to work on time, avoiding lines, avoiding heavy traffic, etc. You do not use probability models to make these decisions; they are made on the basis of past experience that you have been able to process subconsciously, and you let your intuition guide your actions.

In this section the laws of probability are presented. The laws, or rules as they are sometimes called, are very basic but have application in reliability and in quality.

3.2. Laws of Probability

The laws to be discussed are the basic law, complementation, the addition law, and the multiplication law.

BASIC LAW: The basic law describes the probability of a single event:

$$\text{Probability that event A will occur} = \frac{\text{\# of Ways event A can occur}}{\text{\# of Ways event A can occur or fail to occur}} \quad (3.1)$$

It is much easier to write these modes using symbols to represent what the words say. For this model, the following symbols are used:

$P(A)$ = Probability that event A will occur,

m = the # of ways event A can occur,

n = the # of ways event A can fail to occur

When these symbols are put into the basic law model, we get:

$$P(A) = \frac{m}{m + n} \quad (3.2)$$

EXAMPLE #1 (BASIC LAW)

A system that has been used for three years has been successful in eighty-four attempts, and failed sixteen times. What is the system reliability?

Here, $m = 84$, and $n = 16$ (i.e., $100 - 84$)

Then Reliability = $84/(84 + 16) = 84/100 = .84$ or 84%

The result can be illustrated graphically on **probability density function** (p.d.f.). For this example the p.d.f. is illustrated in Figure 3.1

The area of the rectangles in Figure 3.1 is proportional to the probability of the event that the rectangle represents. In this example, the area of the Success rectangle is .84 of the total area; the area of the Failure rectangle is .16 of the total area.

The vertical scale of the probability density function is a probability scale, and as such it goes from zero to one. Hence it is possible to also read the probability of an event from the vertical scale.

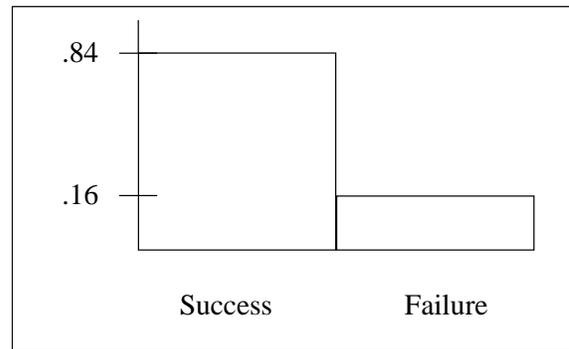


Figure 3.1 Probability Density Function.

When working probability problems it is beneficial to always draw a picture of the situation. It is also useful to shade the area on the p.d.f. that corresponds to the question asked.

3.3. Law of Complementary Events

We have already alluded to this law. It states that the sum of the probability of success and the probability of failure is one. That is:

$$P(S) + P(F) = 1.0 \quad (3.3)$$

If this is true, then we could also state that:

$$P(S) = 1.0 - P(F) \quad (3.4)$$

which is useful when it is easier to compute the probability of failure than it is to compute the probability of success.

3.4. Multiplication Law

The multiplication law can be used to compute the **reliability of a series system**. It is used in reliability prediction when every part or assembly of the system must work for the system to be successful. A reliability block diagram is used to show how the parts (or assemblies) go together for reliability purposes. See Figure 3.2

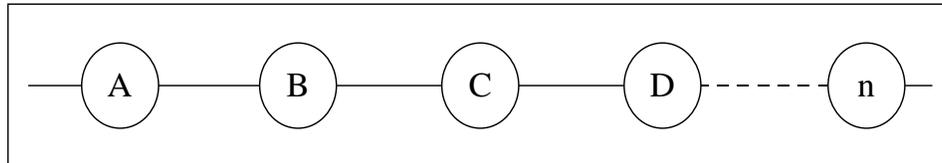


Figure 3.2 Block Diagram for a Series System.

When a block diagram is drawn like this, it means that every block in the system must work to have a success.

The multiplication law breaks down into two cases, for events that are **independent**, and for events that are **conditional** (or not independent).

3.5. Independent Events

An event is independent when the outcome of one event in a series has no affect on the probability of subsequent events. An example of the independent case would be the situation where we have two boxes of different parts made by different vendors. If one part is taken from each box, what is the probability that both parts are good? This is an independent case because the quality of the parts in the first box has no influence on the parts in the second box.

The multiplication law for the independent case is as follows:

Probability that event A is successful and that event B is successful equals the probability of event A times the probability of event B.

When this law is expressed as an equation we end up with:

$$P(A \cap B) = P(A) \times P(B)$$

where " \cap " means "and."

EXAMPLE #2 (MULTIPLICATION LAW - INDEPENDENT EVENTS)

A system has four elements in series. If each element has a probability of success of .99, what is the probability that the system will be successful? See Figure 3.3. The probability for the system is found by computing the product of the four elements, that is:

$$\begin{aligned} P(A \text{ and } B \text{ and } C \text{ and } D) &= P(A) \times P(B) \times P(C) \times P(D) \\ &= .99 \times .99 \times .99 \times .99 \\ &= .9606 \end{aligned}$$

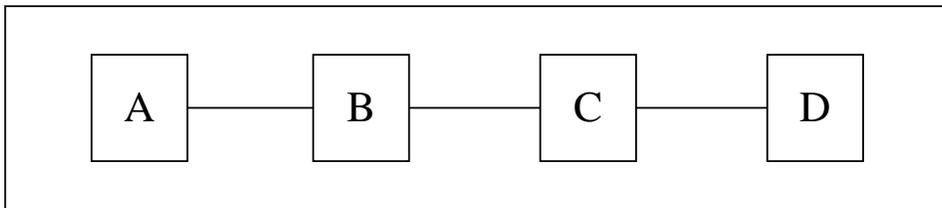


Figure 3.3 Block Diagram for Example # 2.

EXAMPLE #3 (MULTIPLICATION LAW FOR INDEPENDENT EVENTS)

An assembly consists of three sub assemblies which are being selected from three separate boxes. If the process for subassembly A is running at 2% defective, and the process for subassembly B is running at 1% defective, and the process for subassembly C is running at 5% defective, what is the probability of a good system?

We know that $P(A_{\text{FAILS}}) = 2\%$ or .02.

Using the complementation law, the $P(A_{\text{GOOD}}) = 1.00 - .02 = .98$

In the same way, $P(B_{\text{GOOD}}) = .99$, and

$$P(C_{\text{GOOD}}) = .95$$

$$\begin{aligned} \text{For the system, } P(A \text{ and } B \text{ and } C) &= P(A) \times P(B) \times P(C) \\ &= .98 \times .99 \times .95 = \end{aligned}$$

$$= .92169 \approx .92$$

3.6. Conditional Events

Events are conditional when the outcome of one event in a series has an affect on succeeding events in the series. A good example is sampling without replacement from a batch of parts. As each part is drawn from the box, the probability of the next part being good (or bad) changes because the total number in the box is changing and the number of good parts (or bad parts) is also changing.

The multiplication law for the conditional case is as follows:

The probability of events A and B equals the probability of event A times the probability of event B given that event A has already occurred.

When this law is expressed as an equation we end up with:

$$P(A \cap B) = P(A) \times P(B|A)$$

EXAMPLE # 4: A box of 100 parts contains 93 good and 7 defective. If a sample of size 2 is taken from the box, what is the chance that they are both good?

Solution: $P(A) =$ the chance the first is good $= 93/100$
 $= .93$

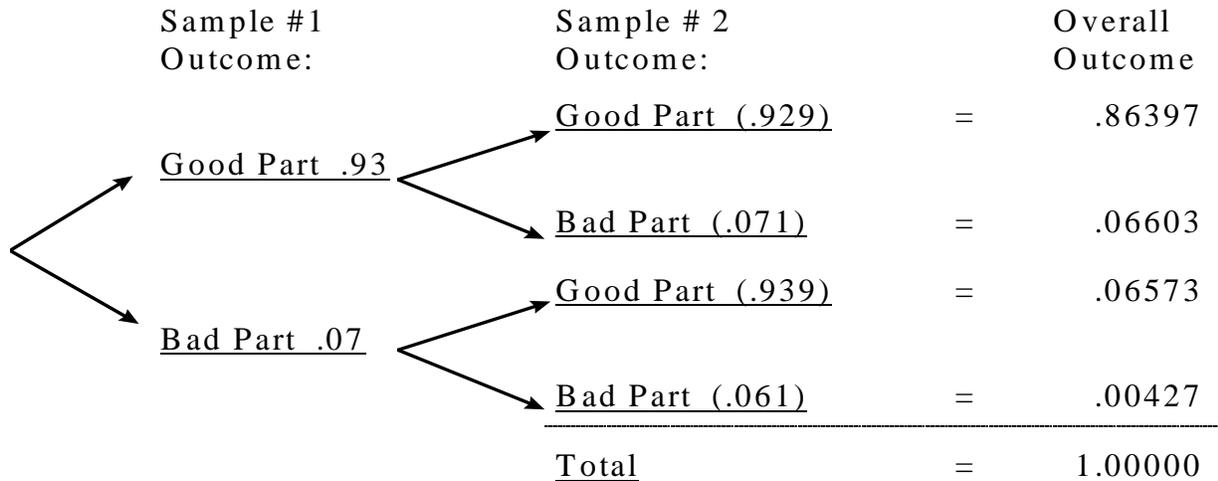
$$P(B/A) = \text{the chance the second is good given that the first was good}$$
$$= 92/99 = .929$$

The chance that both are good is:

$$= P(A) \times P(B/A)$$
$$= .93 \times .929 = .864$$

3.7. Tree Diagrams

An easy way to illustrate the multiplication is a tree diagram. A tree diagram is made up of a series of branches. Each branch represents an outcome of some event. Example 4 would appear as follows on a tree diagram:



3.8. Addition Law

The addition law is used to compute the probability of a single event when more than one outcome is possible. For example, suppose a part is being drawn from a box that contains many parts. If these parts have an inside diameter, then some of the parts could have an inside diameter that is under size, some could be over size, and some could meet the specifications. Hence, three outcomes are possible for the part being drawn, but each part can have only one outcome, i.e., either undersize, oversize, or meeting the specification.

Another example would be the selecting of a piece of M & M candy from a bag of M & Ms. Several colors are possible, but the piece you choose can have only one color.

The addition law breaks down into two conditions: events that are mutually exclusive, and events that are not mutually exclusive.

Events that cannot occur at the same time are mutually exclusive. In the M & M candy example, the colors of the M & Ms are mutually exclusive events because an M & M cannot be more than one color at the same time.

Events that can occur at the same time are not mutually exclusive. For example, when parts from a manufacturing process can have more than one type of defect on the same part, the defects are not mutually exclusive.

The mutually exclusive concept can be illustrated by studying a Venn Diagram.

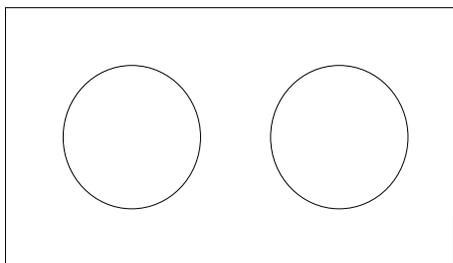


Figure 3.4 Events That are Mutually Exclusive

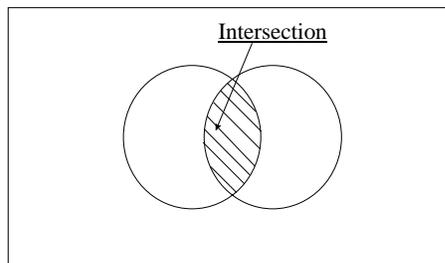


Figure 3.5 Events That Are Not Mutually Exclusive

In Figure 3.4 the circles represent events. Since they do not overlap, the events are mutually exclusive. In Figure 3.5, where they do overlap, the events are not mutually exclusive.

The overlap in Figure 3.5 is called the intersection. This area of the Venn Diagram represents the items in the population that contain both characteristics under consideration.

3.9. Mutually Exclusive Events

The equation used to compute the probability of events that are mutually exclusive is:

$$P(A \cup B) = P(A) + P(B)$$

Probability Law - Mutually Exclusive Events

EXAMPLE # 5. For safety purposes the bottling of aspirins must have a device that shows if someone has tampered with the bottle. Over a six month period, on 3 % of the products shipped, the safety device was damaged, and on .1 % the safety device was missing.

If a bottle is selected at random from the next shipment, what is the probability that the safety device is either damaged or missing?

SOLUTION: Since a bottle cannot have a device missing and be damaged at the same time the addition law for mutually exclusive events applies.

$$P(A) = \text{Probability that the device is damaged,}$$

$$P(A) = .03$$

P(B) = Probability that the device is missing,

P(B) = .001

P(A or B) = P(A) + P(B)

= .03 + .001

= .031

3.10. Non-Mutually Exclusive Events

When events are not mutually exclusive the characteristics being considered can occur at the same time on the item. For example, on M & M candy, the candy could have a flaw on its surface and the color could be the wrong shade. Both of these characteristics could appear on the same piece of candy, therefore these defects are not mutually exclusive.

In manufacturing, parts can have more than one type of defect, which means the defects are not mutually exclusive.

The equation used to compute the probability of events that are not mutually exclusive is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability Law - Not Mutually Exclusive Events

EXAMPLE # 6: Past history on a manufactured part shows that on .003 of the parts the finish is defective; and on .002 of the parts there is a burr that is not acceptable. What is the probability that a part chosen at random has either a defective finish or a burr?

SOLUTION: Since a part could have both defects the addition law for not mutually exclusive events is used:

P(A) = Probability of a defective finish,

P(A) = .003

P(B) = Probability of a burr,

P(B) = .002

P(A or B) = .003 + .002 - (.003 x .002)

= .003 + .002 - .000006

= .005 - .000006

$$= .004994$$

3.11. Computing Probability

To compute the probability of an event it may be helpful to do the following:

1. Identify all the factors related to the problem.
2. If possible draw a picture of the situation, i.e., a Venn Diagram, tree diagram, or other pictorial device that may help you visualize the nature of the situation.
3. If the word "and" is used the multiplication law usually applies; if the word "or" is used the addition law usually applies.

3.12. Hypergeometric Model

The hypergeometric model simplifies the probability calculation for events when the population is finite and sampling is without replacement. It is used to compute the probability of possible sampling results. For example, it can be used to compute the probability of selecting a sample of five items (made up of four good and one defective) from a population of forty (of which thirty are good and ten are defective).

This hypergeometric model involves the use of combination calculations where the number of combinations of "r" things from "n" things is written as:

$$C_r^n = \frac{n!}{r!(n-r)!}$$

where,

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 1$$

"n" represents the number of items that the sample is being taken from; and "r" is the number of items being selected.

NOTE: "n" and "r" are generic terms usually associated with combination models. In the next example different terms are used in the model.

EXAMPLE: From ten parts a sample of three are to be selected. How many different combinations are possible?

$$C_3^{10} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{(10)(9)(8)(7!)}{(3)(2)(1)(7!)} = 120$$

When the hypergeometric model is used to compute the probability of getting a certain sample outcome from a population, the model can be written as follows:

$$P(\text{Event}) = \frac{\left[\begin{array}{l} \text{Number of combinations} \\ \text{of "x" good units in the} \\ \text{sample from the good} \\ \text{units in the population.} \end{array} \right] \left[\begin{array}{l} \text{Number of combinations} \\ \text{of defective units in} \\ \text{sample from defective} \\ \text{units in population.} \end{array} \right]}{\left[\begin{array}{l} \text{Number of combinations} \\ \text{of possible samples from} \\ \text{the population.} \end{array} \right]}$$

where the Event is getting "x" defectives in a sample of "n" units from a population, "N" that contains "D" defectives.

Suppose we let the expressions in the above model be represented by C_1 , C_2 and C_3 . Then,

$$P(\text{Event}) = \frac{C_1 \times C_2}{C_3}$$

The terms used in the hypergeometric are:

n = sample size,

N = population size,

D = number of defectives in the population,

x = number of defectives in the sample.

Using these definitions C_1 , C_2 , and C_3 can be written as:

$$C_1 = \frac{(N-D)!}{(n-x)![(N-D)-(N-X)]!}$$

$$C_2 = \frac{D!}{x!(D-x)!}$$

$$C_3 = \frac{N!}{n!(N-n)!}$$

EXAMPLE: From a box of twenty parts a sample of four is to be selected. If three of the twenty are defective what is the probability of getting exactly three good and one defective in the sample of four?

SOLUTION: We begin by defining the terms:

$$N = 20; \quad n = 4; \quad D = 3; \quad x = 1$$

$$C_1 = \frac{(20-3)!}{(4-1)![(20-3)-(4-1)]!} = \frac{17!}{3!(17-3)!}$$

$$= \frac{17 \times 16 \times 15 \times 14!}{3 \times 2 \times 1 \times 14!} = \frac{17 \times 16 \times 15}{3 \times 2 \times 1} = 680$$

$$C_2 = \frac{3!}{1!(3-1)!} = \frac{3 \times 2!}{1! \times 2!} = 3$$

$$C_3 = \frac{20!}{4!(20-4)!} = \frac{20 \times 19 \times 18 \times 17 \times 16!}{4 \times 3 \times 2 \times 1 \times 16!} = 4845$$

$$P(1 \text{ defective}) = \frac{680 \times 3}{4845} = \frac{2040}{4845} = .421$$

There is a .421 chance of getting exactly one defective in a sample of four from a population of twenty that has three defectives.

DENSITY FUNCTION: The graphical representation of the hypergeometric distribution is illustrated on the hypergeometric density function. The density function shows probability as an

area on a histogram.

For the preceding example where $N = 20$, $n = 4$, and $D = 3$, the density function is as follows:

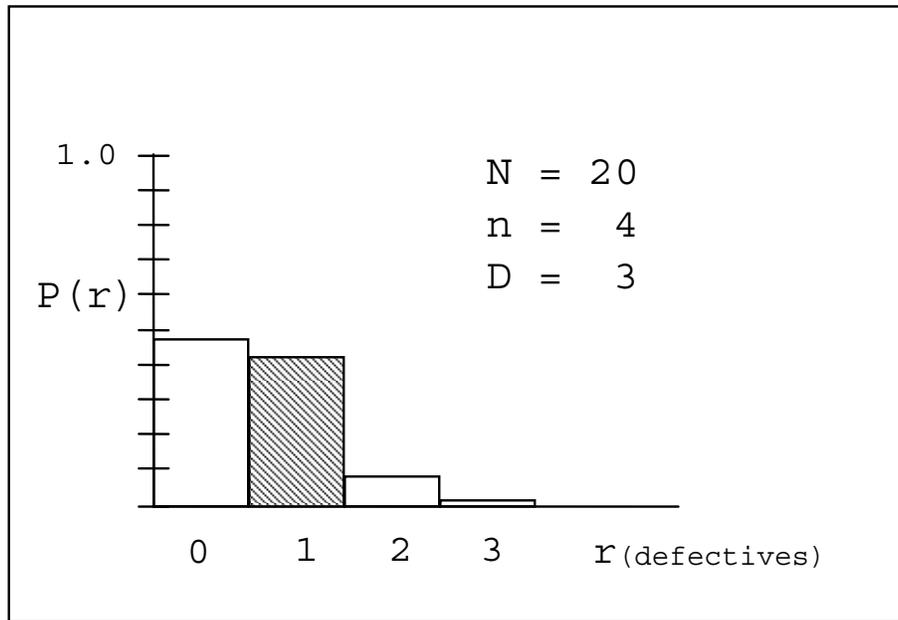


Figure 3.6 Hypergeometric Density Function.

Area on the density function represents probability. For this example, the area of cell "1" shows the probability of 1 defective is .421. The probabilities of 0, 2 and 3 defectives are, respectively, .49128, .0842, and .00359.

3.13. Conditional Probability

Earlier in this section we used the conditional probability model:

$$P(G_1 \text{ and } G_2) = P(G_1) \times P(G_2|G_1)$$

to compute probabilities when certain information was known. One application of this model is sampling without replacement. Suppose you are taking parts from a box one-at-a-time, and not replacing them as they are drawn. Each time a part is taken from the box the population of the box changes. As a result, the probability of all the events following the first event depend on what has been removed from the population, i.e., it depends upon the condition of the population, hence the term, "conditional probability."

The equation above is read, "the probability of getting two good parts in two successive draws without replacement (G_1 and G_2) is equal to the probability the first part is good (G_1) times the

probability the second part is good (G_2) **given that the first part was good (G_1)**."

Example with a die #1: Let us now discuss this concept in a slightly different way. Suppose we have a six sided die numbered 1, 2, 3, 4, 5, 6 and that the sides with numbers 1 and 2 are painted blue (B) and the others are painted green (G), i.e.,

Outcome:	1	2	3	4	5	6
Color:	B	B	G	G	G	G

You already know that the probability of rolling a "4," is 1/6 if all sides are equally likely. However, if I roll the die so you can't see the outcome and then tell you that a green side came up, what probability would you assign to a "4" now? You can forget about the blue sides, "1 and 2," because you know the outcome has to be one of the green sides, "3, or 4, or 5 or 6." To assign probability to a "4," divide the number of ways a "4" can come up by the number of green sides, and you get:

$$P ("4"|G) = \frac{1}{4} = .25$$

since there is one "4" and there are four green sides. The same probability could be assigned to "3, 5, and 6" since they occur at the same frequency as "4."

Note: The probability of .25, or 25%, represents the per cent of the time the result would be a "4" if this experiment (it is given you have a green) were conducted a large number of times.

The possible outcomes for this die are as follows:

1	2	3	4	5	6
B	B	G	G	G	G

in which the following is happening:

1. Numbers "1" and "2" are ignored because they are blue.
2. We are only concerned with the green numbers and there are four of them.
3. One of the green numbers is "4".
4. The probability assigned to "4" is found by dividing the area corresponding to "4" by the total green area.

Example with die #2: Let us now suppose that the die is numbered and colored as follows:

1	1	2	3	4	4
B	B	G	G	G	G

the graph for this die is illustrated below:

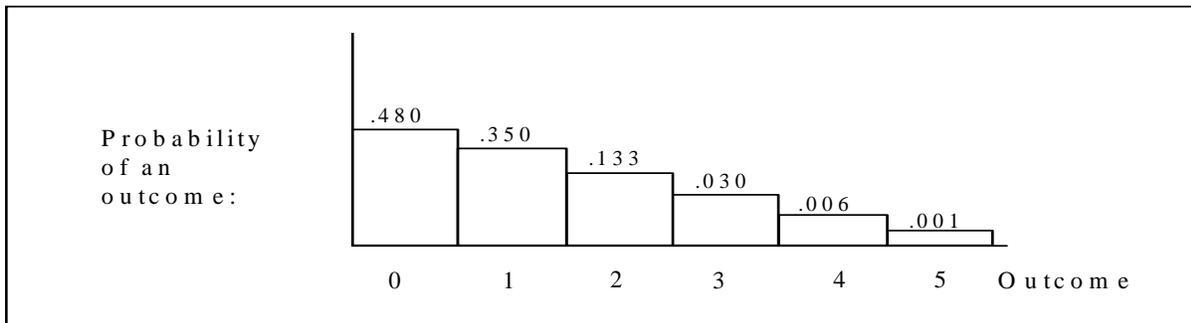
B			G
B	G	G	G
1	2	3	4

The die is rolled and you are told that a green side is showing. The probability you would assign to a "4" is found in the same way: there are two green sides marked "4," and four green sides possible, hence the probability assigned to "4," given that the side is green is .5, i.e.,

$$P("4" | \text{side is green}) = \frac{2 \text{ green sides marked "4"}}{4 \text{ green sides in total}} = .5.$$

For "2|Green" and "3|Green" the probabilities would each be .25 since there is only one green "2" and one green "3."

Discrete density function example: Now let us consider a probability density function in which the probabilities of the possible outcomes have been placed above the cells. We have:



Suppose that these outcomes represent the number of defectives that could be drawn from a population in a sample size of 100. If the question is asked, "What is the probability of getting exactly three defectives in the sample of 100," the answer is .03. But if I told you that I collected the sample of 100 and that there are at least three defectives, what probabilities could be assigned to the outcomes "3, 4, and 5?"

$$P(A \cap B) = P(A) P(B|A) \Rightarrow P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(3 | \text{at least } 3) = \frac{.030}{.030 + .006 + .001} = \frac{.030}{.037} = .811$$

$$P(4 | \text{at least } 3) = \frac{.006}{.037} = .162$$

$$P(5 | \text{at least } 3) = \frac{.001}{.037} = .027$$

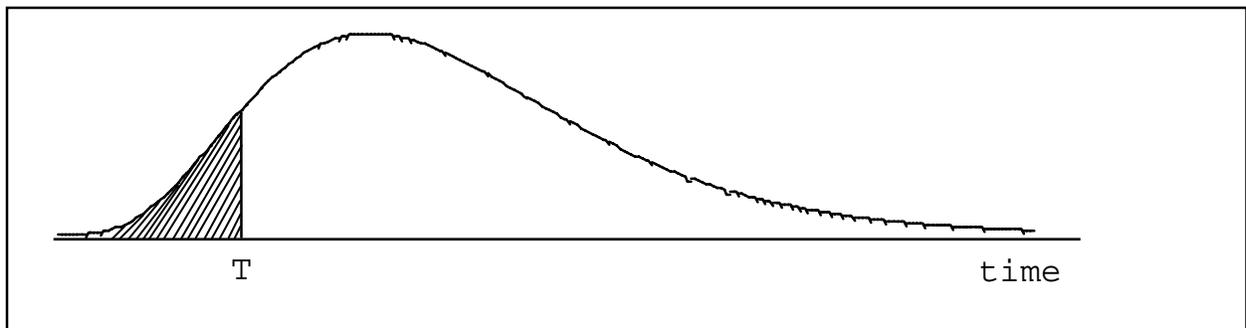
These results tell us that if this sampling were done a large number of times, and when **it is given that you have at least three defectives**, in the sample:

the probability of exactly 3 is:	.811
the probability of exactly 4 is:	.162
the probability of exactly 5 is:	.027

	1.000

Note: This last example is not something a reliability engineer would experience; it is presented here to illustrate the concept of conditional probability because it involves a density function. However, what is likely to be of interest to the reliability engineer is the example that follows.

The same approach is used to calculate these probabilities. A portion of the density function represents what has already happened, and the remaining portion is used to compute the probability desired. Graphically this can be illustrated as follows:



The area to the left of T represents the proportion of the systems that have survived T hours of operation. [Our interest is calculating the probability of a successful mission beginning

at this point in time.]

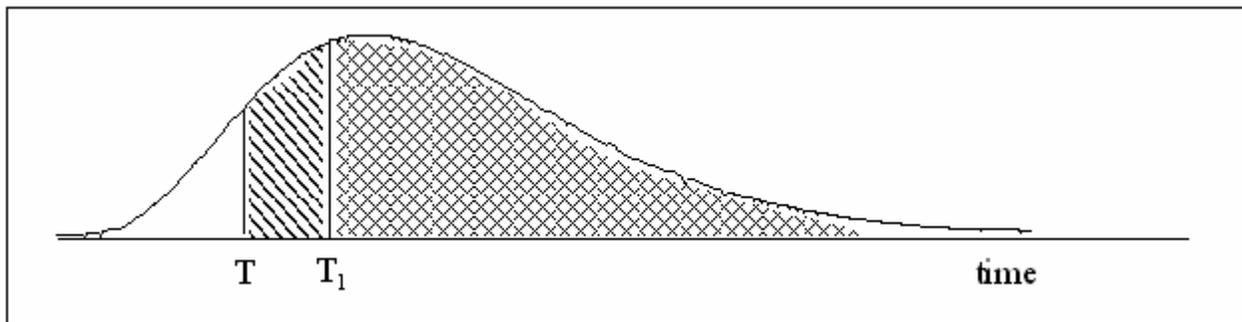
Conditional probability questions are asked as follows: "Given that the system has survived for T hours, what is the probability that a mission of t hours will be successful?" To answer this question, the area from T to T + t is determined; this area is then divided by the area from T to infinity; this gives us the probability of failing in the next "t" hours. The probability of not failing is then subtracted from 1.0 to determine the reliability. In equation form it could be written as follows:

$$\text{Probability of Failure} = \frac{\text{Area under curve corresponding to the mission of } t \text{ hours beginning at } T}{\text{Area under the curve to the right of } T}$$

This gives us the probability of failing during the mission, and the reliability is 1.00 minus that number, or,

$$\text{Mission Reliability} = 1.00 - \text{Probability of Failure.}$$

Graphically it looks like this:



The duration from T to T₁ is the mission time; the area to the right of T is the total chance of failure beyond T.

Computation:

1. Compute the area to the right of T.
2. Compute the area to the right of T₁.
3. The area that corresponds to failure during the mission is the difference between step 1 and step 2.
4. Divide the answer in step 3 by the answer in step 1. (This is the probability of failure.)
5. Subtract the answer in step 4 from 1.00 to find the mission reliability.

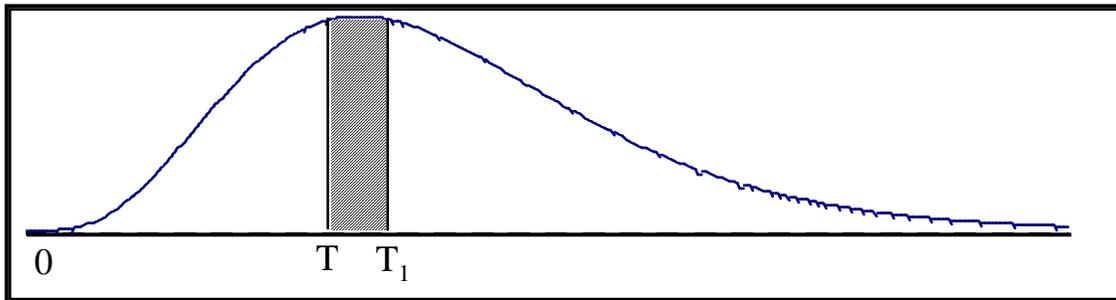
3.14. Summary

This section contains an introduction to basic probability calculations. Six laws were presented:

Basic Law
 Multiplication Law for Independent Events
 Multiplication Law for Conditional Events
 Addition Law for Mutually Exclusive Events
 Addition Law for Not Mutually Exclusive Events
 Complement Law
 Conditional Probability

The laws used most often are the basic law, multiplication law for independent events, addition law for not mutually exclusive events, and the complement law. The application in reliability is primarily in design, prediction, testing, and manufacturing.

Conditional probability will be considered when the exponential, normal and Weibull models are presented. At that time a graph of the density function is used to show how the calculation is made.



The mission is from T to T₁; the area from T to infinity represents the chance of failing after T hours; hence, the chance of failing in the increment from T₁ to T₁ is found by dividing the area between T and T₁ by the area from T to infinity.

The model we will use is as follows:

$$\text{Probability of Failing (during the interval from } T \text{ to } T_1) = \frac{R(T) - R(T_1)}{R(T)} = 1.00 - \frac{R(T_1)}{R(T)}$$

Mission Reliability = 1.00 - P(failing) = $\frac{R(T_1)}{R(T)}$
 (for a mission that begins at T and ends at T₁)

3.15. Equations Used in this Chapter

$$P(A) = \frac{m}{m+n} \quad (3.2)$$

$$P(S) + P(F) = 1.0 \quad (3.3)$$

$$P(A \text{ and } B) = P(A) \times P(B) \text{ (for independent events)}$$

$$P(A \text{ and } B) = P(B/A) \times P(A) \text{ (for conditional events)}$$

$$P(A \text{ or } B) = P(A) + P(B) \text{ (for mutually exclusive events)}$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \text{ (for non-mutually)}$$

$$C_r^n = \frac{n!}{r!(n-r)!}$$

$$P(\text{Event}) = \frac{\left[\begin{array}{l} \text{Number of combinations} \\ \text{of "x" good units in the} \\ \text{sample from the good} \\ \text{units in the population.} \end{array} \right] \left[\begin{array}{l} \text{Number of combinations} \\ \text{of defective units in} \\ \text{sample from defective} \\ \text{units in population.} \end{array} \right]}{\left[\begin{array}{l} \text{Number of combinations} \\ \text{of possible samples from} \\ \text{the population.} \end{array} \right]}$$

$$\text{Mission Reliability} = 1.00 - \text{Probability of Failing} = \frac{R(T_1)}{R(T)}$$

Chapter Four
BINOMIAL DISTRIBUTION

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Symbols, Terms and Definitions

c = the number of occurrences of an event (same as "x" & "r"),
 d = the number of failures (used in the confidence table),
 \ln = natural logarithm, or natural log,
 n = number trials, (can also mean sample size),
 ${}_n C_r$ = the number of combinations of "r" things taken from "n" things,

Note: the combination model is also written as C_r^n

p = probability of an event on a single trial, could also be the fraction defective or the proportion of any characteristic in a population,

\hat{p} = point estimate

q = $1 - p$,

r = the number of occurrences of an event (same as "x" & "c"),

x = the number of occurrences of an event (same as "r" & "c"),

$P(r < 1)$ = probability that r is less than 1,

$P(r \leq 1)$ = probability that r is equal to or less than 1,

$P(r = 1)$ = probability that r equals 1,

$P(r > 1)$ = probability that r is greater than 1,

$P(r \geq 1)$ = probability that r is equal to or greater than 1.

The Binomial Distribution is another discrete distribution in which the observations are always whole numbers. For reliability purposes the observations could be the number of failures in a test, the number of failures in the field, the number of defective parts made on a production line, and so on.

4.1. Introduction

For the Binomial the topics to be covered in this section are:

1. Properties of a Binomial.
2. Parameters of the Binomial.
3. Model used to compute the probability of an event.
4. Binomial Density Function
5. Cumulative probability calculations.
6. How to use Binomial tables to compute probability.
7. Confidence intervals for a Binomial.
8. How to determine the number of trials for the Binomial.

4.2. Properties of a Binomial

It is important to know the properties of a distribution and how to use them because that is how we determine the distribution to use to analyze data. When we work with a Binomial distribution we often think of the outcomes of experiments or sampling opportunities as trials. The properties of the Binomial are:

1. The trials are independent.
2. Only two outcomes are possible for a given trial.
3. The probability of a success remains constant from trial to trial.
4. The trials are identical.
5. In conducting an experiment you can be interested in the number of successes or the number of failures; it depends on which is easier to calculate or find in the Binomial table.

4.3. Parameters of a Binomial

The binomial distribution has two parameters, **n** and **p**. They completely describe the distribution.

RELATIONSHIP TO THE LAWS OF PROBABILITY

Earlier we saw that the hypergeometric distribution can be used to simplify the calculation of a probability when the population is finite and sampling is without replacement.

The binomial can be viewed as a method of computing probability when the population is infinite and the probability of the event does not change. This is the same as sampling with

replacement from a finite population, i.e., when the probability of an event does not change from trial to trial.

EXAMPLE #1: To illustrate, suppose you were sampling a population that consisted of five units of which four are good and one is bad. Let us label these as G1, G2, G3, G4, and B1 to identify the good and bad units.

If a sample of three is selected with replacement, what is the probability of getting exactly two good and one bad? Using the laws of probability the calculation would be as follows:

$$\begin{aligned} P(2 \text{ good}, 1 \text{ bad}) &= P(\text{GGB or GBG or BGG}) \\ &= P(\text{GGB}) + P(\text{GBG}) + P(\text{BGG}) \\ &= (4/5)^2(1/5) + (4/5)^2(1/5) + (1/5)(4/5)^2 \\ &= .128 + .128 + .128 \\ &= .384 \end{aligned}$$

As you can see it was necessary to consider all the possible ways of getting exactly two good and one bad to compute the probability of the event. You should also notice the terms for each way the event could occur have exactly the same factors, two "Gs" and one "B," resulting in 4/5 to appear twice and 1/5 once. In the binomial model the same result will be achieved with a single model. (See example #2.)

4.4. Binomial Model

To compute the probability of an event that fits the binomial, the following model is used:

$$P(x) = C_x^n p^x (1-p)^{n-x}$$

where,

n = number of trials

p = probability of the event on a single trial

x = the number of occurrences of the event in the sampling experiment.

The first term to the right of the equal sign is the symbol used to compute the number of combinations of the event and n and x are as defined.

EXAMPLE # 2. To illustrate the use of the model, consider the problem solved in Example #1, where $n = 3$, $x = 1$, and $p = 1/5 = .2$

$$P(1) = C_1^3 (.2^1) (.8^2) = (3) (.2) (.64) = .384$$

The answer is the same as calculated in Example #1. It should be because the numerical values are the same, i.e., 4/5 or .8 appears twice, 1/5 or .2 appears once, and there are three ways this event can occur.

EXAMPLE # 3. Suppose a lot of 300 parts are 5 percent defective. If a sample of size ten is selected, what is the probability that all the parts in the sample are good?

In this example, $n = 10$, $p = .05$, $x = 0$

$$P(0) = C_0^{10} (.05)^0 (.95)^{10} = (.95)^{10} = .59874$$

4.5. Binomial Density Function

A density function for the hypergeometric distribution was illustrated in the preceding section. For the binomial the density function will look about the same.

The density function has two scales. The horizontal scale represents the possible outcomes of the experiment; the vertical scale is a probability scale and always goes from 0 to 1.0, or 0% to 100%.

The density function for the problem in Example #3 is illustrated Figure 4.1.

The shaded area of the density function depicts the probability of zero for the event.

It is also possible to compute and illustrate on a density function cumulative probabilities. The cumulative probability can be expressed as the probability of "or more" or "x or less."

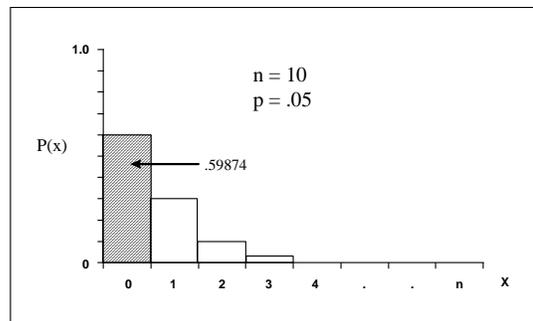


Figure 4.1 Density Function

4.6. Cumulative Probability

Computing the cumulative probabilities of an event requires the summation of probabilities on the left side of the density function or the right side of the density function.

If the need is to find the probability of "x or less" then the left side is cumulated. For example, if you want to calculate the probability of "one or less" you need to add together the probability of zero to the probability of one. In symbols it would look like this:

$$P(x \leq 1) = P(0) + P(1)$$

EXAMPLE #4: Using the data in Example #3, i.e.,

$$n = 10, \quad p = .05$$

P(0)	=	.599	To find the probability of one
P(1)	=	.315	or less, add these two values.
P(2)	=	.074	
P(3)	=	.011	
P(4)	<	.001	

For all other events for this example the probability is less than .001 and has not been listed.

To find the probability of " $x \leq 1$ " we add the values .599 and .315 to get .914. On the density function this can be shown by shading the area above $x = 0$ and $x = 1$. See Figure 4.2.

The cumulative probability can also be illustrated on a cumulative distribution function (CDF). See Figure 4.3. The CDF has application for other distributions also but will not be shown.

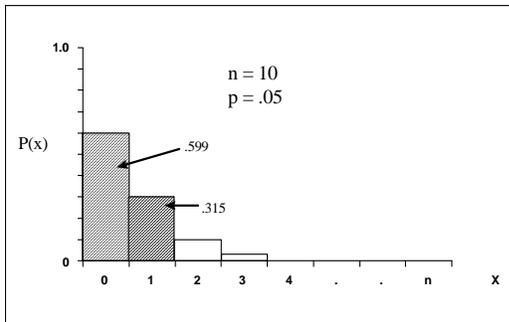


Figure 4.2 Cumulative Probability.

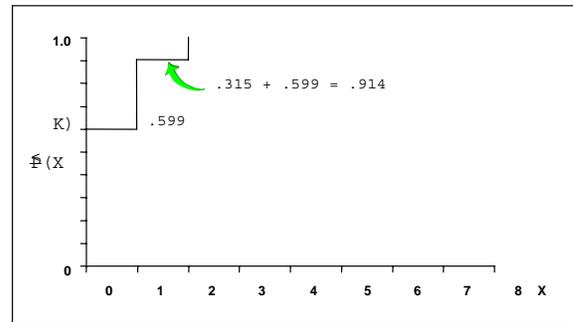


Figure 4.3 CDF for Binomial.

If the probability of an event on the right side of the density function is desired the calculation can be made in a similar way. That is, the right side can be cumulated.

In some cases it is easier to calculate the area on the left side and then subtract that value from one. For example, suppose you want to compute the probability that " $x > 2$." To make this calculation you could add the probabilities of 3, 4, 5, 6, 7, 8, 9, and 10 but that would require the calculation of these eight probabilities and then adding them. It would be easier to make use of the complement law by computing the probability that " $x \leq 2$ " and subtracting the answer from 1.0.

$$P(x > 2) = 1.0 - P(x \leq 2)$$

This equation could also have been written as:

$$P(x \geq 3) = 1.0 - P(x \leq 2)$$

This approach works because:

$$P(0) + P(1) + P(2) + P(3) + P(4) + \dots + P(10) = 1.0,$$

and all we are doing is moving the first three terms on the left side of the equation to the right side of the equation, i.e.,

$$P(3) + P(4) + \dots + P(10) = 1.0 - [P(0) + P(1) + P(2)]$$

4.7. Binomial Table

If a binomial table is available it is much easier to calculate binomial probabilities using the table than using the equation. However, tables are limited because they will not contain all values of p that are possible.

To use the table:

1. Find the table with the correct sample size, n .
2. Find the column with the correct p .
3. Locate the x (or r) value in the left hand column.
4. Where the x row and the p column intersect, read the probability.

If you are using the individual or exact table, the probability provided by the table is the probability of "exactly x " defective.

If you are using the cumulative table the probability provided by the table is the probability of " x or less" defective. However, most binomial cumulative tables cumulate the left side of the density function. Care must be taken in computing right side probabilities.

Note: Binomial tables for cumulative probabilities in other text books may be different; care must be taken when they are used.

EXAMPLE # 5: The problem in Example # 3 can be worked using the binomial table. To find the probability of exactly one defective in a sample of ten when p is .05, look in the exact table for the block headed by $n = 10$, and the column $p = .05$, then come down the r column to $r = 0$ to find $[P(0)]$, i.e., the probability of zero and you should read .599. [Note: In both the exact and cumulative table the $P(0)$ is the same.]

EXAMPLE # 6: Let us now work the problem in Example # 4 using the cumulative table. To find the $P(x \leq 1)$ either table can be used. If the exact table is used, look for the block $n = 10$, the column $p = .05$ and the rows $r = 0$ and $r = 1$. The probabilities for those two rows are then added to determine the probability of one or less, which is .914, (.599 + .315).

If the cumulative table is used the same procedure is used to find the block, column and row. In this case that is block 10, column .05, and row 1. The probability in row 1 is the cumulative probability and is the sum of the probabilities for zero and one. The table should show .914; however, rounding errors sometimes will cause a difference if the calculation is done both ways.

EXAMPLE # 7: If the $P(x \geq 3)$ is desired, it would be easier to calculate the $P(x \leq 2)$ and subtract that value from 1.00 because otherwise you would have to sum the probabilities of 3 through 10. It can be written as follows:

$$\begin{aligned}
P(x \geq 3) &= 1.00 - P(x \leq 2) \\
&= 1.00 - [P(0) + P(1) + P(2)] \\
&= 1.00 - (.599 + .315 + .074) \\
&= 1.00 - .988 \\
&= .012
\end{aligned}$$

4.8. Confidence Intervals

The calculations so far presented have assumed that the probability of an event, p , was known, or assumed. For sampling problems and developing operating characteristic curves this is what must be done because we are asking, "What if..." questions.

In practical applications you are often trying to estimate the location of the population mean, p . This can be done in two ways:

1. **Point Estimate.** You can estimate the mean of the population. The symbol that I will use for the estimate is \hat{p} to distinguish it from the population mean p . This estimate is very simple to calculate:

$$\hat{p} = \frac{\text{Total number of failures}}{\text{Total number of observations}}$$

The advantage of a point estimate is that it is very easy to calculate; the disadvantage is that you do not know how close \hat{p} is to p .

2. **Interval Estimate.** For this estimate a range is calculated. This range is called the confidence interval. The end points of the confidence interval are called the confidence limits.

When a confidence interval has been calculated a confidence statement can be made. The statement is: THE TRUE POPULATION PARAMETER IS SOMEWHERE IN THIS INTERVAL. The statement is made with a predetermined level of confidence.

You can control the likelihood that your confidence statements will be correct by the level of confidence you choose.

Before a confidence interval is calculated you must make certain decisions. You must:

- a. Choose a sample size.

- b. Choose a confidence level (per cent of the times you will be right), or choose a risk level, α , (per cent of the time you are willing to be wrong.)
- c. Decide if you want the confidence interval to be one-sided or two-sided. For a one-sided interval your statement will be that the parameter is **at least** a certain value, or **at most** a certain value. You choose one of these two cases before you collect the data. [For reliability estimates we use the lower confidence limit so we can be sure of **at least** a certain reliability.]

A two-sided interval provides you with both an upper and lower confidence limit; you can then state that the parameter falls between the limits calculated.

Graphically, a two-sided intervals looks like this:

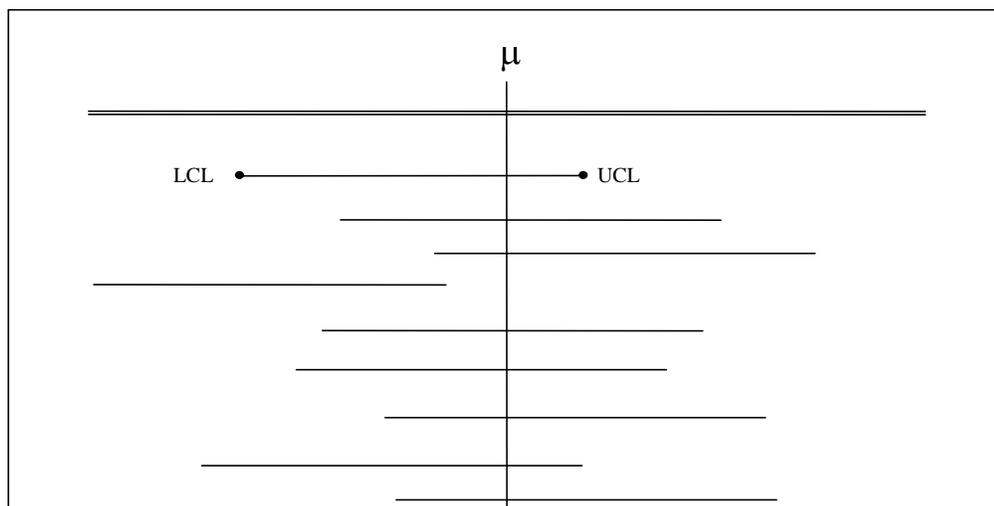


Figure 4.4 Illustrations of Confidence Intervals.

In Figure 4.4, the vertical line in the center represents the parameter being estimated. Note: It would be unknown but is shown here to illustrate the concept.

The horizontal lines represent different samples taken to estimate the parameter. They vary in location because they are random samples. Most of them intersect the parameter, and they represent the times when the statement is correct; the lines that do not intersect the parameter represent the times when the statement is wrong. By letting the risk of a wrong statement be small, most of the lines should cross the center line.

If one hundred interval estimates with ninety per cent confidence were made from the same population, you would expect

ninety of the intervals to intersect the true population parameter and ten would miss the parameter. So, when a confidence statement is made, you are stating the per cent of the time such statements would be correct if many estimates were made. However, when you make a statement on the basis of one sampling, you make the statement as if you are correct.

4.9. Calculating the Interval

The confidence interval can be calculated in several ways:

- A. From confidence tables for a binomial.
- B. From a graph of confidence belts.
- C. From the cumulative binomial tables.
- D. Using a Normal approximation.
- E. Using a Poisson approximation.
- F. Using the F distribution.

METHOD A: Method A is by far the easiest because all one needs to do is to look in the confidence table for a binomial for the appropriate sample size, failure number, and desired confidence.

Two tables are provided in the Appendix of the Portfolio. One table (approximately p. 130 in Appendix of Portfolio) provides a one-sided lower confidence limit for reliability, and the second table provides two-sided limits for unreliability. Unreliability is the same as p for a binomial; reliability then is $1 - p$. In these tables the number of failures is d , and the sample size (number of trials) is n .

It is easy to convert from reliability to unreliability or vice-versa; it is done simply by subtracting either value from 1.00. Hence, if $p = .04$, then reliability is .96; if reliability = .994, $p = .006$. We will use the letter R to signify reliability. Therefore,

$$R = 1 - p \text{ and } p = 1 - R$$

EXAMPLE # 8: To illustrate the use of Method A, suppose that on a test of twenty-two door openers, none of them fail. For 90% confidence (Appendix of Portfolio, approximately p. 134) the reliability is .901. This is found by looking for the block $n = 22$, the column 90% and the row $d = 0$. At the intersection we find the lower limit for reliability: .901.

The statement can be made that the population from which the

sample was taken has a reliability of at least .901; this statement is made with 90% confidence.

It could also be stated with 90 per cent confidence that the unreliability, or probability of failure, is no worse than .099, i.e., $1.000 - .901$.

EXAMPLE # 9: To show how Method A is used with the two-sided confidence table, find $n = 22$, for 90 % confidence and $d = 0$, and you will find two numbers, they are the upper and lower confidence limits. The limits are 0 and .116 but these are for the probability of failure (unreliability). Two statements are possible:

The probability of failure for the population the sample was drawn from is between 0 and .116; or,

the reliability for the population the sample was drawn from is between .884 and 1.000, i.e., $(1.000 - .116;$ and $1.000 - 0)$.

Both statements are made with 90% confidence.

Confidence limits are useful in reliability because they can be used to determine if a specification has been met. For example, suppose that you are working with a specification that calls for a reliability of .90 with a confidence of 90%. If a test is conducted and the lower confidence limit is .917 reliability, which exceeds .90 reliability, then the equipment is considered to have met the specification; but if the lower confidence limit is .873, which falls below .90, the specification has not been met.

METHOD B Method B uses a set of curves that are graphical representations of the cumulative binomial distribution. To use the graphs you proceed as follows using the confidence belts in Figure 4.5. (Dixon, W. J., and Massey, F. J., Introduction to Statistical Analysis, McGraw Hill Inc., N. Y., 1957, Pg. 416.)

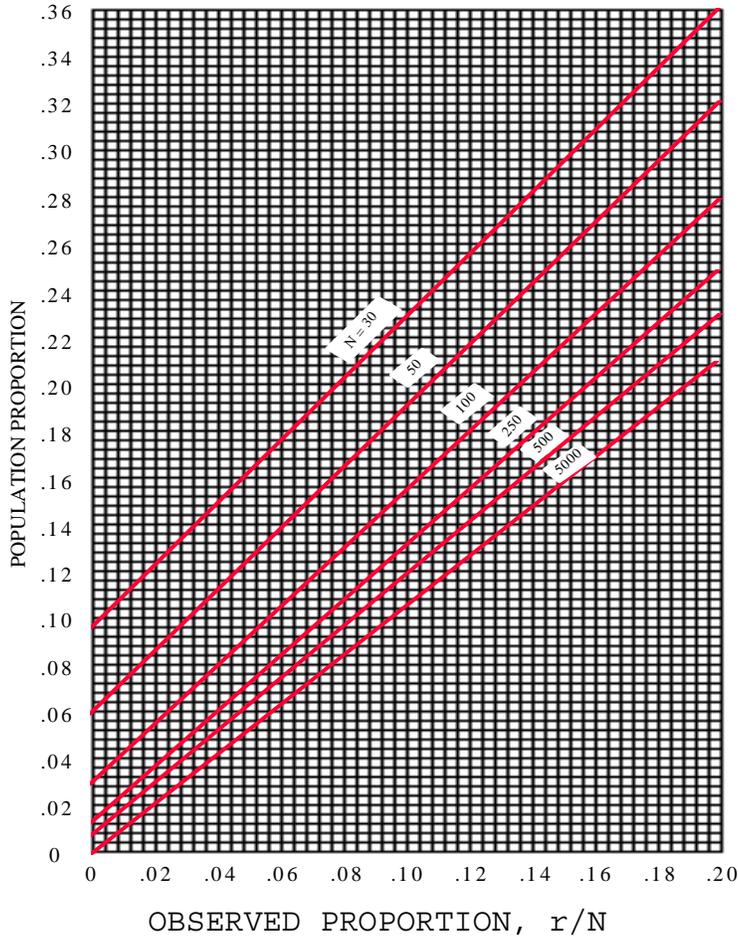


Figure 4.5 One-sided 95% Confidence Limits for a Proportion.

1. Choose the sample size.
2. Collect the failure data, e.g., conduct a test.
3. Count the number of failures and divide by n to determine the "observed proportion," \hat{p} , the point estimate.
4. Enter the bottom scale of the graph with \hat{p} .
5. Read up to the confidence belt that equals the sample size used, or interpolate if the sample size used falls between two of the confidence belts.
6. From the confidence belt go horizontally to the vertical scale on the left side of the graph and read the upper limit for unreliability.
7. To find the lower limit for reliability subtract this number from 1.00.

EXAMPLE # 10: Suppose that in a test of 50 radios 5 fail. To find the 95% lower confidence limit for reliability:

- compute \hat{p} , which is 5/50 or .10,
- find .10 on the 95% graph,
- go up to the confidence belt for $n = 50$,
- go horizontally to the left and read about .20,
- this is the upper limit for the probability of failure,
- the lower limit for reliability is $1.000 - .20$ or .80,
- it can be said with 95% confidence that the reliability of the population the sample was drawn from is at least .80 or 80 %.

METHOD C: Method C involves solving for a set of p values (p_1 and p_2) when the number of trials (n) are known, the number of occurrences (k) are known, and the confidence level or risk (α) is known.

The equations are used to find p_1 and p_2 such that:

1. $P(r \geq (k-1) | n, p = p_1) = 1 - .5\alpha$ (lower confidence limit for p)
2. $P(r \leq k | n, p = p_2) = .5\alpha$ (upper confidence limit for p)

The difficulty with this method is that you need a good set of binomial tables.

Note: The equations apply only to the tables in the Portfolio; if you look in other Binomial tables the equations must be modified. This is because most tables give you "r or more" probabilities; these tables give you "r or less."

Example: In 25 trials 2 failures are found. To find the 95% confidence limits proceed as follows:

1. Find the cumulative table for $n = 25$ (approximately p. 108 in Appendix of Portfolio).
2. Determine α , $\alpha = 1 - \text{confidence}$, or $1 - .95 = .05$; but since these are two-sided limits, we must look for an $\alpha/2$ of .025 and $1 - \alpha/2$ or .975 (as implied by equations 1 and 2 above).
3. For the Upper Confidence Limit find .025 in the row $r = 2$. However .025 is not in the table, but you will find .00896 and .03211 within which .025 falls, i.e.,

$$\begin{aligned} \text{for } p = .25, P(2) &= .03211 \\ \text{for } p = .30, P(2) &= .00896 \end{aligned}$$

If you interpolate between $p = .25$ and $p = .30$ you find that p_2 is about .265. So the upper confidence limit for p is .265.

CUMULATIVE BINOMIAL TABLE FOR $n = 25$ and $p =$													
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.1	0.15	0.2	0.25	0.3
r													
0	0.77782	0.60346	0.46697	0.36040	0.27739	0.21291	0.21291	0.12436	0.07179	0.01720	0.00378	0.00075	0.00013
1	0.97424	0.91135	0.82804	0.73581	0.64238	0.55266	0.55266	0.39472	0.27121	0.09307	0.02739	0.00702	0.00157
2	0.99805	0.98676	0.96204	0.92352	0.87289	0.81289	0.81289	0.67683	0.53709	0.25374	0.09823	0.03211	0.00896
3	0.99989	0.99855	0.99381	0.98348	0.96591	0.94024	0.94024	0.86491	0.76359	0.47112	0.23399	0.09621	0.03324
4	1.00000	0.99988	0.99922	0.99722	0.99284	0.98495	0.98495	0.95486	0.90201	0.68211	0.42067	0.21374	0.9047
5	1.00000	0.99999	0.99992	0.99962	0.99879	0.99694	0.99694	0.98771	0.96660	0.83848	0.61669	0.37828	0.19349

4. For the Lower Confidence Limit find .975 in the row "r - 1," until you find .975 (this is $1 - \alpha/2$), i.e.,

$$\begin{aligned} \text{for } p = .01, P(1) &= .97424 \\ \text{for } p = 0, P(1) &= 1.00000 \end{aligned}$$

If you interpolate between .01 and zero you find p_1 is about .0098. Hence the lower confidence is almost .01.

5. Make a statement: the true proportion is between .01 and .26, this statement is made with 95% confidence.

6. Confidence limits for reliability can be found by subtracting the confidence limits for a proportion from 1.00. In this example the reliability is between .74 and .99.

Note: To find the one-sided lower limit for reliability find p_2 using α , then subtract the answer from 1.00. In the row for $r = k = 2$, look for $\alpha = .05$. Find values .09823 and .03211, which bracket .05. Thus p is bracketed by 0.2 and 0.25, let's say 0.24. Then, the one-sided lower confidence limit for reliability is 76%.

METHOD D: The normal approximation method can be used when the number of trials is large and the proportion is .10 or more. This is to ensure that the distribution is relatively symmetrical.

The equations used are as follows:

Lower confidence limit (LCL) for p :

$$\hat{p} - Z_{\alpha} \sqrt{[\hat{p}(1 - \hat{p}) / n]}$$

Upper confidence limit (UCL) for p :

$$\hat{p} + Z_{\alpha} \sqrt{[\hat{p}(1 - \hat{p}) / n]}$$

where \hat{p} is the proportion of failures in the number of trials, and Z = the standard normal deviate.

Example: Suppose in 25 trials, 2 failures are observed.

$$\hat{p} = 2/25 = .08, \text{ if } \alpha = .05, \text{ then } Z = 1.96, \text{ and } n = 25$$

$$\text{LCL} = .08 - (1.96) \sqrt{[(.08)(.92)/25]} = .08 - .106 = 0$$

$$\text{UCL} = .08 + (1.96) \sqrt{[(.08)(.92)/25]} = .08 + .106 = .186$$

As you can see we have different answers than when the Binomial table is used, this is because the number of trials was small and \hat{p} was large. Hence, unless you have a large number of trials, this method is not recommended.

METHOD E: Poisson approximation. The Poisson can be used to approximate Binomial probabilities when the number of trials is large (at least 10) and when \hat{p} is small (less than .1).

To use this method you proceed as follows:

1. To get the UCL you must find the np value from the Poisson table for the number of failures observed and $\alpha/2$; then divide np by n to determine the UCL.
2. To get the LCL you must find the np value from the Poisson table for the number of failures observed minus one and the $(1 - \alpha/2)$; then divide np by n to determine the LCL.

Example: In 25 trials 2 failures are observed. The 95% confidence limits for p are:

1. Enter the cumulative Poisson table for $r = 2$.
2. Proceed down that column until you come to a probability of .025, (this is $\alpha/2$).
3. You will find .025 in the np row of 7.2.
4. Divide 7.2 by the number of trials, 25, to get .288, which is the upper confidence limit for p .
5. Enter the cumulative Poisson table for $r = 2 - 1$ or 1.
6. Proceed down that column until you come to a probability of .975, (this is $1 - \alpha/2$).
7. You will find .975 in the np row of .24.
8. Divide .24 by the number of trials, 25, to get .0096, or about .01, which is the lower confidence limit for p .

The confidence limits found by this method, .01 and .29 are better estimates than we found with the Normal approximation method. These estimates would be even better if the n were larger and \hat{p} were smaller; these are also much easier to use.

METHOD F: Using the F table. This method can be found in Kapur and Lamberson, "Reliability in Engineering Design," pg. 379. They are called the Non-Bayesian and Exact Confidence Limits for the Binomial.

The upper limit is found using the equation:

$$UCL = \frac{(r+1)F_{\alpha/2, 2(r+1), 2(n-r)}}{(n-r) + (r+1)F_{\alpha/2, 2(r+1), 2(n-r)}}$$

$$LCL = \frac{r}{r + (n-r+1)F_{\alpha/2, 2(n-r+1), 2(r)}}$$

Example: In 25 trials 2 failures are observed, the 95% confidence limits are:

$$r = 2, n = 25, \alpha = .05$$

$$UCL = \frac{(3)F_{.025, 6, 46}}{23 + (3)F_{.025, 6, 46}} = \frac{(3)(2.71)}{23 + (3)(2.71)} = \frac{8.13}{31.13} = .261$$

$$LCL = \frac{2}{2 + (24)F_{.025, 48, 4}} = \frac{2}{2 + (24)(8.407)} = \frac{2}{203.8} = .0098$$

From this example you can see that this method gives us the best approximation to the Binomial since .01 and .26 are what was obtained using the Binomial table. However, you do need an F table to use this method.

To show how these methods compare, a summary of the results obtained from five of these methods follows:

Method:	A ¹	C	D	E	F
LCL:	.014	.0096	0	.0096	.0098
UCL:	.238	.26	.186	.288	.261

1: See portfolio appendix table 21, p.143 two-sided confidence limits for a proportion.

4.10. Developing a Sampling Plan

You have already been exposed to one method for developing a discrete sampling plan. The one-sided confidence limit table is an excellent way to determine the number of trials and acceptance number for testing a system that fits the binomial distribution.

To determine the plan you need only to:

1. Decide on the reliability value that is to be demonstrated. It may come from a specification,
2. Decide on a confidence level. It may come from a specification.
3. Select an acceptance number. Note: An acceptance number of zero will minimize the number of trials but can have adverse affects on other risks.
4. Find a number of trials in the table by searching the table for the acceptance number and reliability desired in the confidence level column that applies.

The drawback to this approach is that the tables only go up to $n = 30$ which eliminates high reliability requirements.

Another method involves a calculation that uses an acceptance number of zero. The equation is:

$$n = \frac{\ln(1 - \text{confidence})}{\ln(\text{Reliability})} = \text{the number of trials}$$

EXAMPLE # 10: A specification calls for a reliability of .95 to be demonstrated with 90% confidence. How many trials will be required to demonstrate this specification if zero failures are allowed?

Solution: The confidence is 90% or .90; the reliability is .95; the number of failures is 0, the number of trials is:

$$n = \frac{\ln(1 - .90)}{\ln(.95)} = \frac{\ln .1}{\ln .95} = \frac{-2.30259}{-.05129} = 44.9 \approx 45$$

Note: If the equipment being tested has a constant failure rate, a sample unit can be used for more than one trial of the experiment.

Sampling plans for acceptance numbers other than zero can be found in the chapter on Testing.

4.11. Summary

This chapter contains information on how to deal with data

that fits the binomial model. This is useful distribution because it has many applications in reliability and quality control.

In reliability, the binomial is used to compute reliability for systems where success is based on the number of successful trials; the binomial can also be used to compute the confidence interval for this type of system; and the binomial can be used to determine the number of trials required to demonstrate a given reliability with a prescribed confidence level.

Trials and sample sizes have been used through out this chapter. When sample size is used, it is understood that each sample represents a trial. However, in reliability testing where the trial size is large, a sampling unit can be used for more than one trial if the system failure rate is constant. This is a necessary condition for the binomial model, i.e., that p is constant from trial to trial.

The binomial has many other features that have not been included in this chapter but what is in here serves as an introduction to the binomial.

4.12. Equations

$$\hat{p} = \frac{\text{Total number of failures}}{\text{Total number of observations}}$$

$$P(r) = {}_n C_r p^r (1 - p)^{n-r}$$

$$P(r \leq x) = P(0) + P(1) + P(2) + \dots + P(x)$$

$$P(r \geq x) = P(x) + P(x + 1) + P(x + 2) + \dots + P(n)$$

$$P(r < x) = P(0) + P(x + 1) + P(x + 2) + \dots + P(x - 1)$$

$$P(r > x) = P(x + 1) + P(x + 2) + \dots + P(n)$$

$$n = \frac{\ln(1 - \text{confidence})}{\ln(\text{Reliability})} = \text{the number of trials (0 fail.)}$$

$$P(r \geq (k-1) | n, p = p_1) = 1 - .5\alpha \text{ (lower confidence limit for } p)$$

$$P(r \leq k | n, p = p_2) = .5\alpha \text{ (upper confidence limit for } p)$$

Lower confidence limit (LCL) for p:

$$\hat{p} - Z_{\alpha} \sqrt{[\hat{p}(1 - \hat{p}) / n]}$$

Upper confidence limit (UCL) for p:

$$\hat{p} + Z_{\alpha} \sqrt{[\hat{p}(1 - \hat{p}) / n]}$$

$$\text{UCL} = \frac{(r + 1)F_{\alpha/2, 2(r + 1), 2(n - r)}}{(n - r) + (r + 1)F_{\alpha/2, 2(r + 1), 2(n - r)}}$$

$$\text{LCL} = \frac{r}{r + (n - r + 1)F_{\alpha/2, 2(n - r + 1), 2r}}$$

Chapter 5

Poisson Distribution

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SYMBOLS USED IN THIS SECTION:

$e = 2.71828+$, the base of the natural log system,
 m = the expected number of occurrences of an event,
 np = the expected number of occurrences of an event when
the Poisson is used as an approximation to the Binomial,
 $P(r)$ = the probability that an event will occur r times
 r = the number of occurrences of an event,

5.1. Introduction

This distribution is named after Simeon Dennis Poisson who lived from 1781 to 1840. His family tried to make him into everything from a surgeon to a lawyer, the last on the theory that he was fit for nothing better. He decided to become a mathematician when he found he could solve a problem involving a jug of wine. [Two friends have an eight-quart jug of wine and want to share it evenly. They also have two empty jars, one holds five quarts and the other holds three quarts.]¹ (Newman)

5.2 The Poisson as an Approximation to the Binomial

The Poisson distribution has two applications: it can be used to compute answers to binomial problems if the sample size, n , is large and the probability of an event, p , is small; the larger n , and the smaller p , the better the approximation. But what is large and what is small? I have found that when n is at least 10 and p is .1 or smaller, the approximation is useful; however, if $n > 20$ and $p < .05$, the approximation is better. What it really depends upon is the use that is being made of the approximation.

When the Poisson is used in this way, the model is:

$$P(r) = \left(e^{-np} \right) \left(\frac{(np)^r}{r!} \right) \quad (5.1)$$

Example #1. In Example #4 in the Binomial section, a sample of 10 was selected from a process running .05 defective; the probability of no defectives was found to be .59874. Using the Poisson model we get:

$$\begin{aligned} P(0) &= e^{-(10)(.05)} \frac{.5^0}{0!} && (5.2) \\ &= e^{-.5} = .60653 \end{aligned}$$

which differs by .00779 from the binomial calculation. np in this equation is the expected number of defectives in the sample, r is the number of occurrences and e is the base of the natural log system.

5.3 The Poisson as a Distribution in its Own Right

A second application is as a distribution in its own right. In this case the model is:

$$P(r) = e^{-m} \frac{(m)^r}{r!} \quad (5.3)$$

where,

m = the expected number of occurrences of an event, and
 r = the number of occurrences for which a probability is desired.

Here are some examples of situations that fit a Poisson process. (The first six were taken from Grant's, "Statistical Quality Control.")²

- excessive rainstorms
- articles turned in to the Lost and Found
- calls from a coin-box telephone
- deaths from a kick of a horse
- vacancies in the U.S. Supreme Court
- errors in alignment found at aircraft final inspection
- errors on a typed page,
- tornadoes
- accidents of various kinds,
- failures of equipment during a mission,
- expected failures during a test,
- flat tires,
- inventory demands, etc.

These situations fit a Poisson because there is a large number of possibilities for the event described to occur, but the probability that it occurs at a specific point in time (flat tire), or in a specific word (typed page), etc., is very small.

In the "articles turned in ..." example, data has been collected on the number of lost articles turned in to Lost and Found in a large office building. The distribution is as follows:

# of Articles turned in (r):	0	1	2	3	4	5	6	7
Number of days:	169	134	74	32	11	2	0	1

On 169 days no lost articles were turned in, on 134 days one lost article was turned in, etc.

If you think about this problem you will realize that if there are a large number of people using this building the opportunity to lose something and have it turned in could be very large. You should also understand that the number of people using the building must be the about the same from day to day.

5.4. Properties of a Poisson Process

- the distribution can be used to analyze defects (whereas the binomial is used for defectives),
- the possible ways an event can occur must be large,
- the probability of an event occurring must be small,
- trials are independent,
- the distribution is defined by one parameter, m , the expected number of occurrences of an event,
- the sum of the probabilities for all possible events is 1.0

5.5 Parameters of a Poisson

The Poisson has only one parameter, m , the expected number of occurrences of the event. If m is known, the entire distribution can be described.

5.6 Poisson Model

The model which was first given in 1837 is as follows:

$$P(r) = e^{-m} \frac{(m)^r}{r!} \quad (5.4)$$

where, m = the expected number of occurrences of an event,

$e = 2.71828+$ (base of the natural log system),

r = the number of occurrences of the event.

The probability of r occurrences, i.e., $P(r)$, can be computed using a scientific calculator or table like the Cumulative Poisson table in the Appendix of the Portfolio.

5.7 Poisson Table (cumulative):

c→	0	1	2	3	4	5	6	7						
m↓									c→	15	16			
0.82	.440	.802	.950	.990	.998	1.000			5.60	1.000				
0.84	.432	.794	.947	.989	.998	1.000			5.80	1.000				
0.86	.423	.787	.944	.988	.998	1.000			6.00	.999	1.000			
0.88	.415	.780	.940	.988	.998	1.000			6.20	.999	1.000			
0.90	.407	.772	.937	.987	.998	1.000			6.40	.999	1.000			
0.92	.399	.765	.934	.986	.997	1.000			6.60	.999	.999			
0.94	.391	.758	.930	.984	.997	1.000			6.80	.998	.999			
0.96	.383	.750	.927	.983	.997	1.000			7.00	.998	.999			
0.98	.375	.743	.923	.982	.997	.999	1.000		7.20	.997	.999			
1.00	.368	.736	.920	.981	.996	.999	1.000		7.40	.996	.998			
1.10	.333	.699	.900	.974	.995	.999	1.000		7.60	.995	.998			
1.20	.301	.663	.879	.966	.992	.998	1.000		7.80	.993	.997			
1.30	.273	.627	.857	.957	.989	.998	1.000		8.00	.992	.996			
1.40	.247	.592	.833	.946	.986	.997	.999	1.000						
1.50	.223	.558	.809	.934	.981	.996	.999	1.000						
1.60	.202	.525	.783	.921	.976	.994	.999	1.000	8	9	10	11		
1.70	.183	.493	.757	.907	.970	.992	.998	1.000	1.000					
1.80	.165	.463	.731	.891	.964	.990	.997	.999	1.000					
1.90	.150	.434	.704	.875	.956	.987	.997	.999	1.000					
2.00	.135	.406	.677	.857	.947	.983	.995	.999	1.000					

The Poisson table in the Portfolio is a cumulative Poisson; it cumulates the left tail of the Poisson density function from $r = 0$ to whatever you want "r" to be. The body of the table (see table above) contains the cumulative probability of an event. It has two scales, "m" or "np" is found on the vertical scale on the left side of the table, and the cumulative "r" is found on the horizontal scale across the top of the table. To find the probability of an event, find "m" in the vertical and "r" in the horizontal column and read the probability where the column and row intersect.

CUMULATIVE PROBABILITY: A cumulative "r" means "r or less;" the probabilities in the column below that r value represent the sum of the probabilities for that r value and all the r values below it. For example, if $m = 1.5$ and $r \leq 3$, the table value is .934, and this is the sum of the probabilities for 0, 1, 2, and 3.

EXACT PROBABILITIES: If the probability of exactly 3 is desired, i.e., $P(r = 3)$, it is necessary to subtract the $P(r \leq 2)$ from the $P(r \leq 3)$ because the difference between $P(r \leq 2)$ and $P(r \leq 3)$ is $P(r = 3)$ since $P(r \leq 2)$ includes 0, 1, and 2, and $P(r \leq 3)$ includes 0, 1, 2, and 3. A similar calculation is made for other

exact probabilities.

Example #2. In the final inspection at an automobile assembly plant, 12 cars with leaking radiators were discovered last week.

- a. If no corrective action was taken, what is the probability that none of the cars produced this week will have leaking radiators?

Solution: $m = 12$; $r = 0$

$$P(0) = e^{-12} \frac{12^0}{0!} = .000006$$

- b. What is the probability that exactly one of the cars produced in a day will have a leaking radiator?

Solution: Assuming that production is the same, or nearly the same each day, the number of leaking radiators expected each day is found by dividing 12 by 5 (assuming a five day week), or $12/5 = 2.4$. The probability of exactly one leaking radiator is:

$$P(1) = e^{-2.4} \frac{(2.4)^1}{1!} = .21772 \text{ or about } .22$$

- c. Suppose that when more than 3 leaking radiators are found corrective action must be taken. How many days during the month will it be necessary to take corrective action?

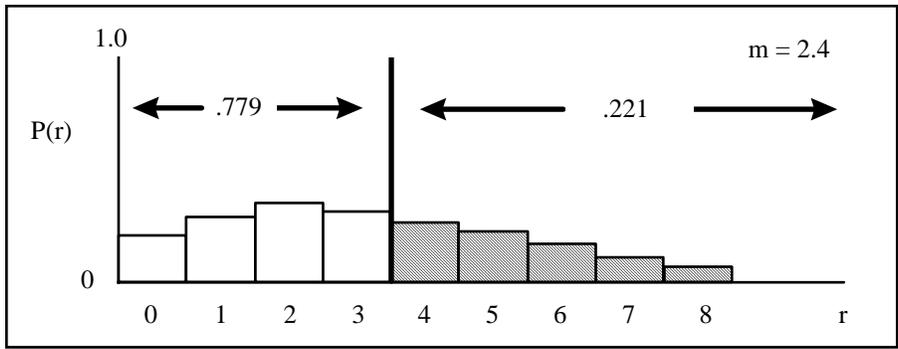
Solution: From the previous problem we know that $m = 2.4$ per day; and for this problem if $r > 3$ corrective action is taken. To determine this probability it will be easier if the complement is computed and then subtracted from one, i.e.,

$$P(r > 3) = 1.0 - P(r \leq 3)$$

$$\begin{aligned} P(r \leq 3) &= P(0) + P(1) + P(2) + P(3) \\ &= .779 \text{ (from table)} \end{aligned}$$

$$P(r > 3) = 1.0 - .779 = .221$$

Therefore, we would expect to have more than 3 leaking radiators on about 22% of the days of the month. If there are 20 working days a month then on $.22 \times 20$ days or about 4.4 days of the month corrective action will be taken. See the illustration:



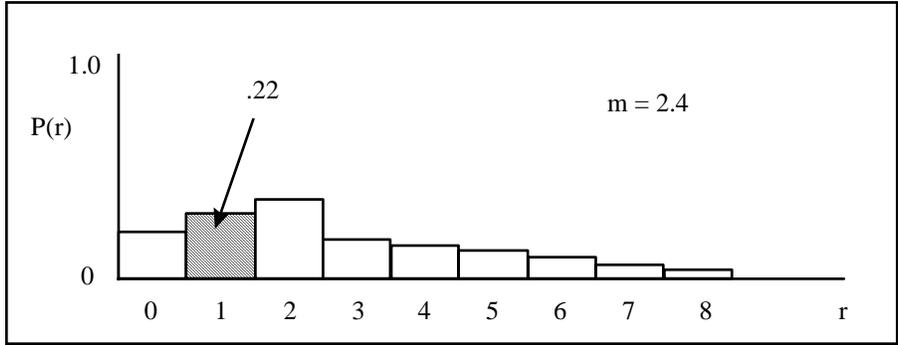
THORNDIKE CURVE: Another source for determining cumulative probabilities for a Poisson distribution is the Thorndike curve. A copy of curve can be found in the Appendix of the Portfolio.

The curve is set up with "m" on the horizontal scale, and "r" as curved lines crossing the body of the graph. The left vertical scale is $P(r \leq x)$. The curve is useful because it contains the entire Poisson cumulative table on one page but it is not as accurate as the table. To see how the curve works try solving the last example with the Thorndike curve.

5.8 Poisson Density Function

The Poisson probability density function shows probability as an area in a series of rectangles. Each rectangle represents the probability of exactly one of the "r" occurrences of the event for a process with a mean equal to "m." The left-hand scale is a probability scale.

The probability density function for the problem in Example #2 is shown below:



The probability density function is constructed by drawing a rectangle corresponding to the probability "r," using the left hand scale to determine the height of the rectangle.

EXAMPLE #3:

A typist has been keeping track of the number of errors made and finds that his error rate is one error every two pages. (This was based on 100 pages that contained 50 errors; since 50/100 is .5, the error rate is .5 per page, or an average of one error on every 2 pages.)

- a. What is the probability that a 5 page letter will contain no errors?

What is known?

$m = .5$ per page;

therefore, on 5 pages we expect $.5 \times 5$ or 2.5 errors

m (for 5 pages) = 2.5

$$P(0) = e^{-2.5} \frac{2.5^0}{0!} = .0825$$

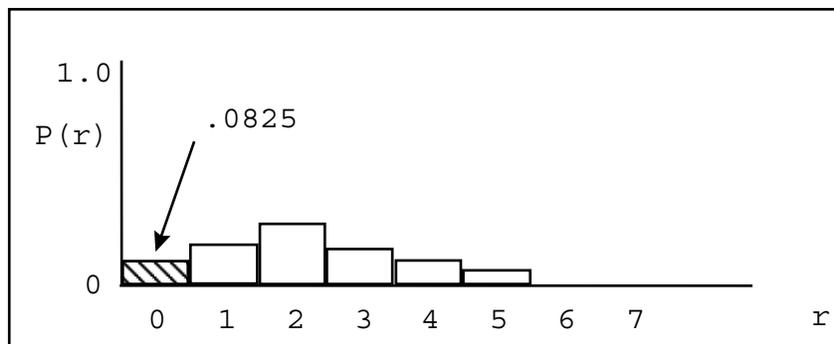
From the table: Using the $m = 2.4$ and $m = 2.6$, the probabilities are averaged for $r = 0$, i.e.,

$$m = 2.4 = .091 \text{ for } r = 0$$

$$m = 2.6 = .074 \text{ for } r = 0$$

 .165 which when divided by 2 equals .0825.

- b. Construct the probability density function for this problem.



Example #4

Inventory demands for a product have averaged 4 per week

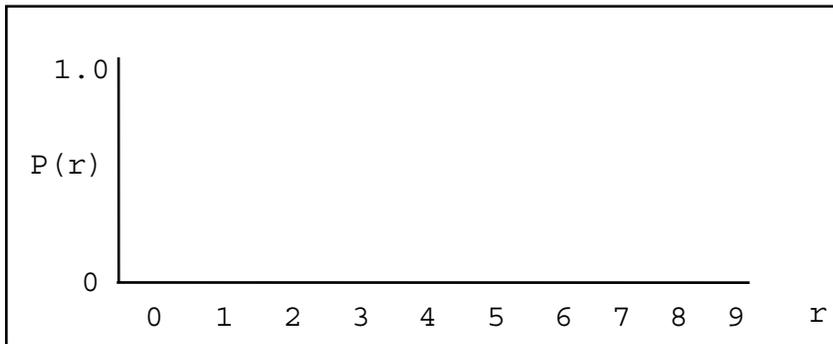
during the past year. If you want to have enough units on hand at the start of each week to be able to cover the demands each week with 90% confidence, how many units should be on hand at the start of each week?

Solution: $m = 4/\text{week}$ $P(r \leq x) = .90$ $x = ?$

The question is, what should "x" be?

To solve this problem find 4 in the "m" column of the table; go across that row until you come to a probability of at least .90; read the column you are in ("x"). The answer is 7.

The student should construct the probability density function for this problem.



5.9 Summary

The Poisson distribution is a very useful distribution. You will see it being used throughout the course for various situations. It is probably the easiest distribution to use to compute probability because the table is easy to use and covers a good range of "m" values.

5.10. Equations

$$P(r) = e^{-np} \frac{(np)^r}{r!} \quad (5.1)$$

$$P(r) = e^{-m} \frac{(m)^r}{r!} \quad (5.3)$$

$$P(r > 3) = 1.0 - P(r \leq 3)$$

$$P(r \leq 3) = P(0) + P(1) + P(2) + P(3)$$

Chapter 6

THE EXPONENTIAL DISTRIBUTION

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Symbols, Terms and Definitions

MTBF	=	Mean time between failure
$R(t)$	=	Reliability for a mission of "t" hours
$h(t)$	=	hazard function (instantaneous failure rate)
θ	=	the mean time between failure (MTBF)
λ	=	the failure rate.
α	=	risk
$\sum t_i$	=	sum of failure times for the units that failed
n	=	sample size
c	=	number of failures
$n-c$	=	number of units that did not fail
T_L	=	time when test was stopped
t	=	mission time
T	=	total test time.
stop on a time	=	stop a test at pre-determined time
stop on a failure	=	stop a test on a pre-determined number of failures
with replacement	=	replace or fix test units as they fail
without replacement	=	do not replace or fix test units as they fail
confidence interval	=	band that, at a given confidence level, contains the parameter being estimated
confidence limits	=	end points of a confidence interval

6.1. Introduction

The exponential distribution is a continuous distribution that can be used to calculate reliability for systems that have a constant failure rate. A system with a constant failure rate is one that has the same reliability, or probability of success, from mission to mission as long as the system is not in the wearout state.

The exponential distribution is used to compute the reliability for electronic equipment; in addition, complex systems can have constant failure rates. The exponential model should not be used for prediction, or other purposes, unless it is known that the system has a constant failure rate. If it is not known, or if there is some doubt about how the failure rate behaves, the failure data should be analyzed.

Failures can be analyzed in various ways: one method is to plot the failure rate over time to see if it is constant but this requires extensive testing or a large amount of field data and both of these may be hard to get. The Kologorov-Smirnov (K-S) can also be used to test how well an exponential fits the data but large samples are required; this is called a goodness-of-fit test. Another approach is to use Weibull Probability Paper to analyze failure data, but as in the case of the K-S test, a large sample is recommended.

6.2. Properties of the Exponential Distribution

- a. It has only one parameter, the Mean Time Between Failure, (or the failure rate).
- b. The MTBF and failure rate are reciprocals, i.e.,
 $1/\text{MTBF} = \text{failure rate}$, and $1/\text{failure rate} = \text{MTBF}$. (6.1)
- c. The mean occurs at the 63rd percentile on the probability density function (p.d.f.).

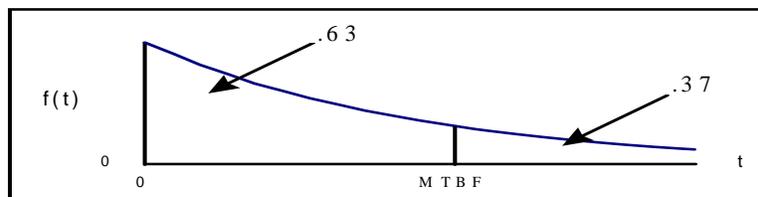


Figure 6.1 Density Function for an Exponential

- d. The density function is defined as follows:

$$f(t) = [1/\text{MTBF}] [e^{(-t/\text{MTBF})}] = \lambda e^{-\lambda t} \quad (6.2)$$

Note: "f(t)" is the height of the density function for a given value of "t."

- e. Mission time is the length of a mission and can occur at any point on the density function's horizontal axis.
- f. The area to the right of the mission time on the p.d.f. is the reliability for that time.

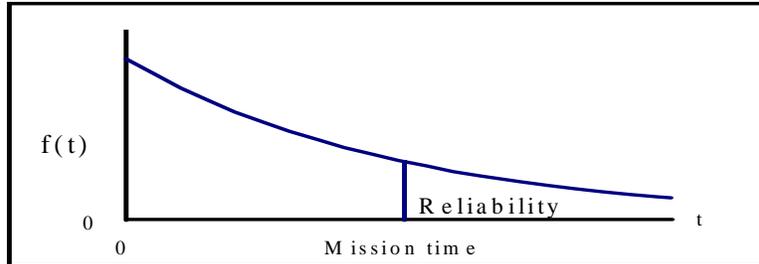


Figure 6.2 Reliability as an Area Under the Density Function Curve.

- g. Reliability is a probability, a number between 0 and 1; a reliability of 0 corresponds to certain failure, and a reliability of 1 means that success is certain.
- h. The reliability function is defined as:

$$R(t) = e^{(-t/MTBF)} = e^{-\lambda t} \quad (6.3)$$

- i. The reliability function shows the reliability for any mission time.

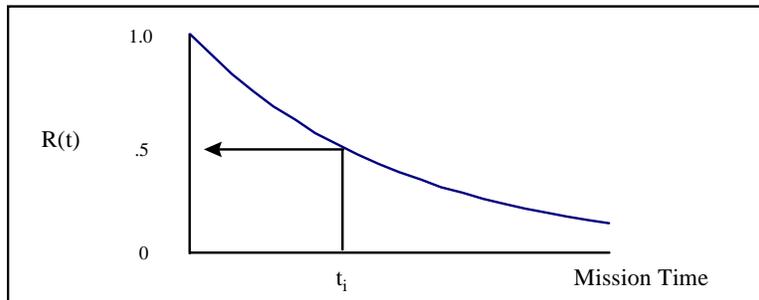
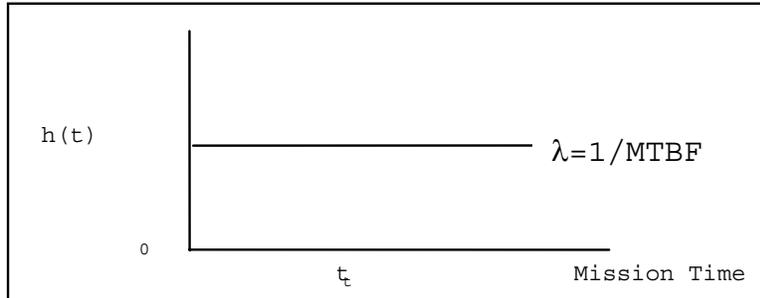


Figure 6.3 Reliability Function for

To find the reliability from the reliability function enter the Mission Time scale at the appropriate point and proceed on a vertical line to the curve, proceed on a horizontal line to the R(t) scale to read the reliability.

- j. The hazard rate $h(t)$, aka the instantaneous failure rate, is constant; it is defined as the density function divided by the reliability function. This property is easy to see if the two functions are examined because everything cancels out except for the failure rate.

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$



Hazard rate function for the exponential case

- k. As mission time decreases for a given MTBF, or as MTBF increases for a given mission time, reliability increases.

6.3. Exponential Reliability Model

The exponential reliability model used to compute reliability can be written in either of the two ways that follow. They give the same answer in both cases so use which ever is easier for the problem you are solving.

$$R(t) = e^{-\lambda t}$$

$$R(t) = e^{-\frac{t}{\theta}}$$

In both cases,

t = mission time

$R(t)$ = the reliability for a mission of " t " hours.

θ = the mean time between failure (MTBF)

λ = the failure rate.

Example #1: For a mission of 3 hours what is the reliability if the MTBF is 60 hours?

Solution: $t = 3$ and $MTBF = 60$

$$R(3) = e^{-3/60} = e^{-.05} = .95$$

A three hour mission has a .95 or 95% chance of success; or 95 out of 100 missions are expected to be successful, and 5 out of 100 or 5% of the missions are expected to fail.

If the mission time is decreased the reliability increases for a given MTBF. If the mission is reduced to 2 hours,

$$R(2) = e^{-2/60} = e^{-.033} = .967$$

If the MTBF increases the reliability increases, for example, what if the mission time is 2 hours and the MTBF is 100 hours,

$$R(2) = e^{-2/100} = e^{-.02} = .98$$

On the probability density function (p.d.f.), these three probabilities would show up as areas to the right of the mission time; as the mission time decreases or the MTBF increases the area under the curve to the right of the mission time increases.

Approximation model: Reliability can also be calculated using an approximation model that gives accurate results to at least the second decimal place if the exponent of the "e" in the reliability model is .1 or less. The calculation is as follows:

$$R(t) = 1 - t/\theta \tag{6.6}$$

For example, $R(2) = e^{-.02} = 1 - .02$ or $.98$

This equation can also be used to find the exponent of "e" if the reliability is known:

$$t/\theta = 1 - R(t) \tag{6.7}$$

For example, if $R(4) = .94$, then $t/\theta = 1.0 - .94 = .06$, then the MTBF or mission time can be determined, e.g.:

if $t/\theta = .06$, and

$t = 4$, then,

$$\theta = 4/.06 = 66.7 \text{ hours.}$$

This approximation works because the Reliability function for the exponential is close to being linear for reliability

values near 1.00. The expansion of $\exp(-x)$ also illustrates why this works:

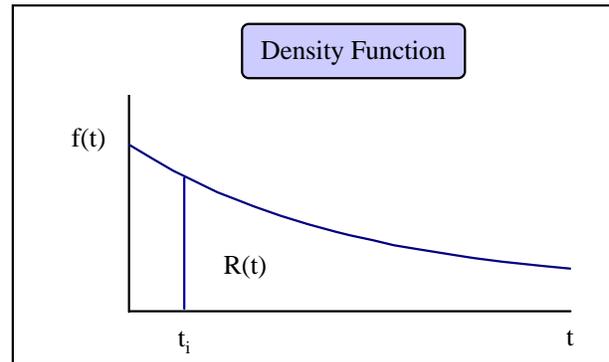
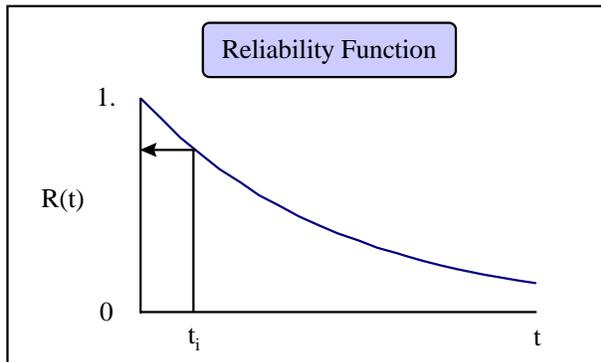
$$e^{-x} = 1 - x^1/1! + x^2/2! - x^3/3! + x^4/4! - \dots$$

6.4. The Density Function and the Reliability Function

The density function is useful for certain mathematical operations, but for this course it is used primarily to illustrate that reliability is an area under the density curve.

The reliability function is useful if the curve is drawn accurately on a large sheet of graph paper because it can then be used to estimate reliability for any mission time. This is done by entering the horizontal scale at the mission time, proceeding on a vertical line to the curve and then moving directly to the left on a horizontal line to read the reliability.

Reliability, which can also be determined from the density function as an area, is a point on the vertical scale of the reliability function. Hence, the reliability function can be thought of as a graph that can be used to estimate the reliability for an infinite number of density functions.



6.5. The Hazard Function

The Hazard function shows the instantaneous failure rate for a system. For the exponential the failure rate is a horizontal line and is independent of time. If the $f(t)$ is divided by the $R(t)$ the result is a constant, the failure rate (or $1/\text{MTBF}$). This means that the failure rate is not a function of time so long as the equipment is not in the wearout phase or the infant mortality phase as depicted on a bathtub curve.

6.6 Estimates of MTBF

The situation commonly faced by reliability engineers is how to estimate the MTBF, a reliability characteristic, of the population being sampled. You might want to know the MTBF so you could estimate how many spares are going to be needed or to see

if a certain requirement is being met.

Two estimates are possible, a point estimate and an interval estimate.

POINT ESTIMATE: The point estimate for the MTBF is found by dividing the hours accumulated on the system by the number of failures.

$$\hat{\theta} = \text{Estimated MTBF} = \frac{\text{Total Test Time}}{\text{Total Number of Failures}}$$

However, when the MTBF is being estimated for test data the denominator of the equation can vary depending on the situation. For example, in testing, tests can be stopped on a failure or at a predetermined time. If a test is stopped at a predetermined time and no failures have occurred then an MTBF cannot be estimated since the denominator would be zero. To avoid this possibility, "1" is added to the number of failures. This results in a more conservative estimate of the MTBF and as a result is not accepted by all users. If such a model is to be used it is crucial that both parties, the buyer and the seller, agree to this practice before testing begins. Research has shown that it would be better to add .5 or .25 but at this time "1" is commonly added to the number of failures.

Another factor in testing is whether the test is run with replacement or without replacement. When tests are run with replacement, units that fail are either replaced (or repaired) and the test continues; when the test is run without replacement, units are not replaced as they fail. In both cases a total test time is needed but it is calculated in different ways.

With Replacement - To find the total test time when testing is with replacement the number of units on test (n) is multiplied by the time when the test was stopped (T_L).

Without Replacement - The total test time when testing is without replacement is the sum of two times, the total time for those that failed and total time for those that did not fail. That is, the total time for the units that failed ($\sum t_i$) is first computed, then added to the total time for the units that did not fail ($(n - c) \times T_L$).

The consequence of these conditions is that there are four different possibilities related to estimating the MTBF from test data:

- Stop at a time - with replacement
- Stop at a time - without replacement
- Stop on a failure - with replacement
- Stop on a failure - without replacement

The models that go with each of these possibilities are presented in the table that follows:

<u>ESTIMATING MTBF, $\hat{\theta}$</u>		
TEST IS	TEST IS STOPPED ON A FAILURE	TEST IS STOPPED ON TIME
Without Replacement	$\frac{\sum t_i + (n - c)T_L}{c}$ CASE I	$\frac{\sum t_i + (n - c)T_L}{c + 1}$ CASE II
With Replacement	$\frac{nT_L}{c}$ CASE III	$\frac{nT_L}{c + 1}$ CASE IV

where,

$\sum t_i$ = sum of failure times for the units that failed
 n = sample size
 c = number of failures
 $n - c$ = number of units that did not fail
 T_L = time when test was stopped

Example #2: A test of 30 units is run for 40 hours each without replacement; if 2 failures are observed at 10 and 23 hours, what is the sample MTBF?

Solution: This is a test without replacement and stopping on a time; this is case II on the matrix above, therefore,

$$\begin{aligned}
 n &= 30, & \sum t_i &= 10 + 23 = 33, & c &= 2, \\
 c+1 &= 3, & n &= 30, & T_L &= 40, & n - c &= 30 - 2 = 28
 \end{aligned}$$

$$\hat{\theta} = \frac{33 + 28(40)}{3} = \frac{33 + 1120}{3} = \frac{1153}{3} = 384.3 \text{ hours}$$

Example #3 A test of 10 units with replacement has been conducted. The test was stopped on the 5th failure with failure times of: 5, 14, 27, 50, 100. What is the sample MTBF and what is the reliability for a 4 hour mission for a system with an MTBF as determined from this test?

Solution: This is a test with replacement, stopping on the 5th failure which is case III, therefore,

$n = 10,$ $T_L = 100,$ $c = 5,$ resulting in an MTBF of:

$$\hat{\theta} = \frac{10(100)}{5} = 200 \text{ hours}$$

and the reliability for a 4 hour mission is:

$$R(4) = e^{-4/200} = e^{-.02} = 1 - .02 = .98$$

Point estimates are easy to calculate but do not provide a measure of how close they are to the true mean, i.e., the population mean. A better approach would be to compute a confidence interval.

CONFIDENCE INTERVALS and CONFIDENCE LIMITS: The confidence interval is an estimate of the range within which the parameter being estimated is located; the end points of this interval are called confidence limits. LCL is the lower confidence limit and UCL is the upper confidence limit.

The advantage of a confidence interval estimate over a point estimate is that the confidence interval estimate provides you with a measure of accuracy. Hence, you know how close the estimate may lie to the true parameter being estimated. E.g., if the confidence limits for the MTBF are 40 and 48 hours, the interval width is 8 hours, which is more accurate than an estimate with an interval of, say, 50.

CONFIDENCE STATEMENTS: Once the interval is calculated, a statement can be made about the MTBF, with a predetermined level of confidence $(1 - \alpha)\%$. The statement is either right or wrong because the interval either includes the parameter or it does not include the parameter. If calculations like this are made repeatedly the corresponding statements will be correct $(1 - \alpha)\%$ of the time.

In making statements, it is not correct to say, "there is a 95% probability that the MTBF is between 40 and 48." It is more correct to say: "The true MTBF is between 40 and 48; this statement is made with 95% confidence." The latter statement implies that when such statements are made they will be correct 95% of the time.

Risk (α) - When confidence statements are made there is some

risk involved. There is a risk that the confidence statement is incorrect, i.e., that it does not include the true parameter being estimated. Risk is the complement of the confidence level, if the confidence level is .90 then the risk is .10 or 10 per

cent. This means that 90% of the statements made are correct and 10% are wrong. Mathematically,

$$\text{Confidence} = 1 - \alpha \quad (6.8)$$

$$1 - \text{Confidence} = \alpha = \text{risk} \quad (6.9)$$

If you decrease the risk you increase the confidence and you widen the confidence interval. The result is that you are allowing for more sampling error. So it usually is not advantageous to make the risk too low because wider limits are less useful. On the other hand, if the confidence level is lowered, the risk increases and the confidence limits get narrower, but then you may not have sufficient confidence in what is being estimated. The confidence levels used most often are 90% or 95% because they are high enough to give you a reasonable amount of confidence, and by controlling the number of failures observed, the resulting confidence interval can be made acceptable.

ONE-SIDED or TWO-SIDED - You have the choice of computing one-sided or two-sided limits. Two-sided limits are used when there is interest in how low or how high the parameter could be. But when confidence limits are used to determine if a reliability requirement or an MTBF has been met, the lower confidence is all that is needed. If the lower confidence limit is higher than the requirement, the requirement has been met, otherwise, it has not been met. The choice between one-sided or two-sided is usually related to the purpose for making the computation.

CHI-SQUARE DISTRIBUTION: If a system's failures follow the exponential failure model, averages of samples drawn from that exponential population fit a Chi-Square (χ^2) Distribution.

It can be shown that:

$$2 \times (\text{number of failures}) \frac{\text{sample MTBF}}{\text{population MTBF}} \quad (6.10)$$

has a χ^2 Distribution with $2 \times (\text{number of failures})$ degrees of freedom when the test is terminated on a failure.

In other words,

$$\frac{2r (\text{sample MTBF})}{(\text{population MTBF})} = \frac{2r\hat{\theta}}{\theta} = \chi^2 \quad (6.11)$$

where,

$r = c = \text{number of failures}$ (Both are used)
 $\theta = \text{the population MTBF}$

But,

$$(r \times \text{sample MTBF}) = \text{total test time} \quad (6.12)$$

since the sample MTBF is found by dividing the total test time by the number of failures when the test terminates on a failure.

Hence, the equation (6.11) can be written as:

$$\frac{2(\text{TotalTestTime})}{\theta} = \chi^2 \quad (6.13)$$

Since the population MTBF is what we are estimating, we can re-write this as:

$$\frac{2(\text{TotalTestTime})}{\chi^2} = \theta \quad (6.14)$$

or:

$$\theta = \frac{2T}{\chi^2} \quad (6.15)$$

where,

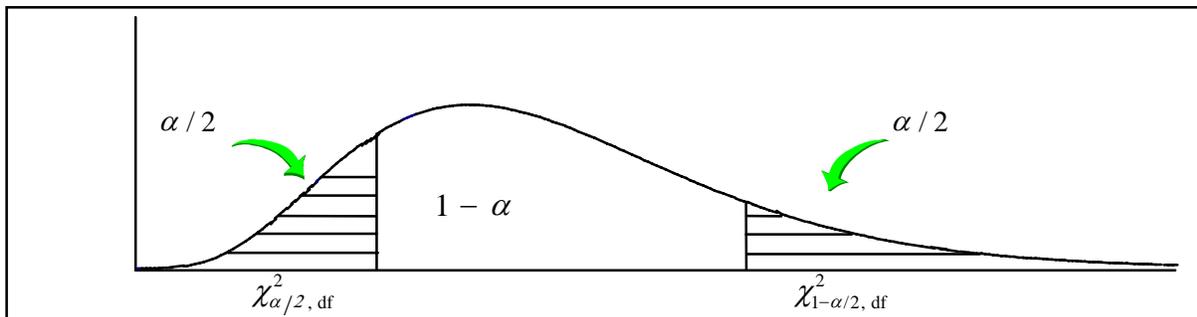
$$T = \text{Total Test Time} \quad (6.16)$$

In place of the "=" sign the inequality sign is used, " \leq or \geq " so that we can speak of an interval rather than a specific point.

For two-sided limits, and stopping on a failure, we end up with the limits:

$$\frac{2T}{\chi^2} \leq \theta \leq \frac{2T}{\chi^2} \quad (6.17)$$

However, the confidence level has not been attached to the Chi-Square values. On a two-sided interval the confidence is divided equally between the two tails of the distribution. On a Chi-Square distribution the confidence level can be illustrated in the following way:



where,

$\chi_{\alpha/2,df}^2$ = the Chi-Square value for the lower side of a Chi-Square distribution with a certain number of degrees of freedom, and

The $1-\alpha/2$ part

$\chi_{1-\alpha/2,df}^2$ = the Chi-Square value for the upper side of a Chi-Square distribution with a certain number of degrees of freedom.

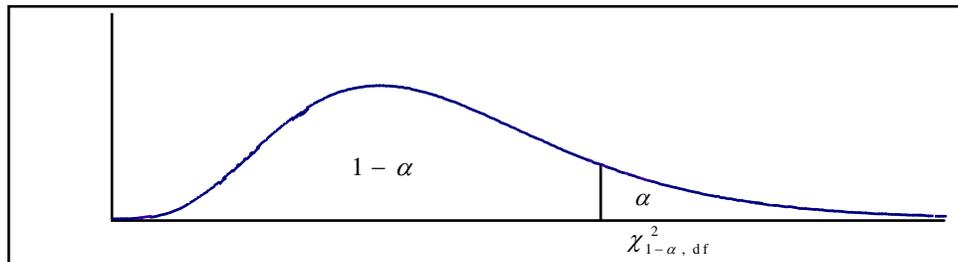
The χ^2 value is found in the Chi-Square table using the subscripts attached to Chi-Square. "1 - α " identifies the column you need and "df" identifies the row you need, where

$$df = 2r, \text{ if the test is stopped on a failure, and } \quad (6.18)$$

$$df = 2r + 2 \text{ if the test is stopped on a time.} \quad (6.19)$$

Note: $2r + 2$ is derived from $2(r + 1)$

For a one-sided test all the risk (α) is in one tail, and for reliability it is the upper tail. The Chi-Square distribution for this situation is as follows:



Putting this all together we end up with four models:

MODEL I. Two-sided limits stopping on a failure:

$$\frac{2T}{\chi_{1-\alpha/2,2r}^2} \leq \theta \leq \frac{2T}{\chi_{\alpha/2,2r}^2} \quad (6.20)$$

MODEL II. Two-sided limits stopping on a time:

$$\frac{2T}{\chi_{1-\alpha/2,2r+2}^2} \leq \theta \leq \frac{2T}{\chi_{\alpha/2,2r}^2} \quad (6.21)$$

Note: Only the lower limit has a df of $2r + 2$; the upper limit does not because $2r$ is more conservative than $2r + 2$ at the upper limit, while $2r + 2$ is more conservative than $2r$ at the lower limit. Stopping on a time leads to a more conservative estimate,

because "1" is added to the number of failures for the lower limit.

MODEL III. One-sided lower limit stopping on a failure:

$$\theta \geq \frac{2T}{\chi_{1-\alpha, 2r}^2} \quad (6.22)$$

MODEL IV. One-sided lower limit stopping on a time:

$$\theta \geq \frac{2T}{\chi_{1-\alpha, 2r+2}^2} \quad (6.23)$$

Note: One-sided upper limits are not given because there is less interest in the upper limits for reliability and MTBF.

CHI-SQUARE TABLE: The Chi-Square table provides Chi-Square values for confidence levels from .5% to 99.5%, and for degrees of freedom from 2 to 50. To find the appropriate Chi-Square value, locate the column (confidence level) and the row (df); where they intersect the Chi-Square value is found. For example if you wanted the 95% confidence level for 16 degrees of freedom, look for the 95% column and the 16th row; where they intersect the value is 26.3. See table below:

30.0	40.0	50.0	60.0	70.0	80.0	90.0	95.0	97.5	99.0	99.5
.148	.275	.455	.708	1.07	1.04	2.71	3.84	5.02	6.63	7.88
.713	1.02	1.39	1.83	2.41	3.22	4.61	5.99	7.38	9.21	19.8
1.42	1.87	2.37	2.95	3.67	4.04	6.25	7.81	9.35	11.3	12.8
21.9	2.75	3.36	4.04	4.88	5.99	7.78	9.49	11.1	13.3	14.9
3.00	3.66	4.35	5.13	6.06	7.29	9.24	11.1	12.8	15.1	16.7
3.83	4.57	5.35	6.21	7.23	8.56	10.6	12.6	14.4	16.8	18.5
4.67	5.49	6.35	7.28	8.38	9.80	12.0	14.1	16.0	18.5	20.3
5.53	6.42	7.34	8.35	9.52	11.0	13.4	15.5	17.5	20.1	22.0
6.39	7.36	8.34	9.41	10.7	12.2	14.7	16.9	19.0	21.7	23.6
7.27	7.27	9.34	10.5	11.8	13.4	16.0	18.3	20.5	23.2	25.2
8.15	9.24	10.3	11.5	12.9	14.6	17.3	19.7	21.9	24.7	26.8
9.03	10.2	11.3	12.6	14.0	15.8	18.5	21.0	23.3	26.2	28.3
9.93	11.1	12.3	13.6	15.1	17.0	19.8	22.4	24.7	27.7	29.8
10.8	12.1	13.3	14.7	16.2	18.2	21.1	23.7	26.1	29.1	31.3
11.7	13.0	14.3	15.7	17.3	19.3	22.3	25.0	27.5	30.6	32.8
12.6	14.0	15.3	16.8	18.4	20.5	23.5	26.3	28.8	32.0	34.3
13.5	14.9	16.3	17.8	19.5	21.6	24.8	27.6	30.2	33.4	35.7
14.4	15.9	17.3	18.9	20.6	22.8	26.0	28.9	31.5	34.8	37.2
15.4	16.9	18.3	19.9	21.7	23.9	27.2	30.1	32.9	36.2	38.6

Example #4: What is the lower 90% confidence limit for the MTBF for the data in Example #2 (A test of 30 units ran for 40 hours without replacement; 2 failures were observed at 10 and 23 hours)?

Solution: In this problem the total test time is $33 + 28(40)$ or 1153 hours, and the test is stopped on a time, hence,

$$T = 1153, r = 2, \text{ and } df = 2r + 2 = 2(2) + 2 = 6 \\ \text{(stopped on a time),}$$

$$\chi_{.90, 6}^2 = 10.6$$

$$\theta \geq \frac{2(1153)}{10.6} = \frac{2306}{10.6} = 217.6 \text{ hours}$$

Statement: The true MTBF is at least 217.6; this statement is made with 90% confidence.

Note: If you conducted this test many times, you would be correct in your statements 90% of the time and wrong 10% of the time.

If you review the answer to Example #2 you will see that the point estimate for the MTBF is 384.3, which is considerably higher because 384.3 is an estimate of the population mean and therefore close to the center of the distribution. The confidence for a point estimate is between 50% and 60% for a Chi-Square, as illustrated below:

$$\text{The lower 50\% confidence limit} = \frac{2306}{5.35} = 431.0$$

$$\text{The lower 60\% confidence limit} = \frac{2306}{6.21} = 371.3$$

6.7 Confidence Statements for Reliability

To make confidence statements for reliability compute the $R(t)$ but use the lower confidence limit for the MTBF in place of the MTBF in the denominator of the exponent, i.e.,

$$R(t)_{\text{conf}} = e^{-t/\text{LCL}}$$

In the previous example, if a reliability confidence statement were to be made for a 10 hour mission,

$R(10)_{90\% \text{ conf}} = e^{-10/217.6} = e^{-.0460} = .9550$, i.e., we are 90% confident that the reliability for a 10-hour mission is at least 95.5%.

$$\text{whereas } R(10) = e^{-10/384.3} = .974 = 97.4\%.$$

6.8 Summary

The primary focus of this chapter was on how the exponential distribution is used to calculate reliability and confidence limits. Also included was the calculation for the hazard function and point estimates.

Chapter 7

Reliability Allocation

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SYMBOLS AND MODELS USED IN THIS SECTION:

- n = the number of subsystems,
- R_i = the reliability of subsystem i ,
- R_{sys} = the reliability of the system.
- λ_i = subsystem failure rate,
- t_i = mission time for the i th subsystem.
- n_i = the number of modules in the i th unit,
- $R(t)$ = the reliability requirement for the system,
- I_i = the importance factor for the i th unit,
- N = the total number of modules in the system.

7.1. Introduction

Reliability allocation is the process of breaking down the reliability requirement for a system into reliability requirements for the units that make up the system.

7.2. Equal Allocation Model

EXAMPLE #1: If a series system with two subsystems has a reliability requirement of .99, what reliability must each subsystem have so that their product is not less than .99? The answer can be found by taking the square root of .99, which is .99498744 or about .995. (As a check $.995 \times .995 = .990025$.) Therefore, each subsystem needs a reliability of .995 for the system to have a reliability of at least .99. [Note: There are many other answers possible, in fact, every pair of numbers whose product is at least .99 is a solution.]

The reason for taking the square root to allocate the reliability is related to the multiplication law for probability. To find the probability that a series of independent units operate successfully, their probabilities are multiplied, for example,

$$P(A \text{ and } B) = P(A) \times P(B) \quad (7.1)$$

In allocation this process is reversed; if we already know what is required, that is, the product of A and B, then finding values for A and B is accomplished by taking the square root of the product. If there were three units in series the cube root would be taken, and for "n" units in series the nth root is taken.

This is called the equal allocation model, and it can be written as follows:

$$R_i = (R_{\text{sys}})^{1/n} \quad (7.2)$$

where,

- n = the number of subsystems in series,
- R_i = the reliability allocated to each subsystem,
- R_{sys} = the reliability requirement for the system.

The equal allocation model is easy to apply and its primary purpose is to provide the user with a starting place. The next step is to see if the allocation is attainable.

CHECKING ATTAINABILITY: In the first problem presented, an allocation of .995 was obtained using the equal allocation model, that is, subsystem A is assigned a reliability of .995 and subsystem B is also assigned a reliability of .995. Can subsystems A and B achieve a reliability of .995? What if subsystem A can do no better than .992? What reliability would be needed by subsystem B so that the system requirement of .99 is met? We know that:

$$R_A \times R_B = .99, \text{ and,}$$

$$.992 \times R_B = .99, \text{ therefore,}$$

$$R_B = .99/.992 = .99798387 \text{ or about } .998.$$

As a check, $.998 \times .992 = .990016$.

Another point to consider is the cost, development time, criticality and other aspects of the units in the system. When some subsystems are more critical, have longer development times, or have failure rates that are more expensive to obtain, it is important that such information be considered in the allocation process.

FAILURE RATE: When the subsystem reliability has been determined, a failure rate can easily be calculated for subsystems that fail exponentially. The subsystem failure rate is found by taking the natural log of the subsystem reliability and dividing by the mission time, that is,

$$\ln [R(t_i)] = (-t_i) (\lambda_i)$$

$$\lambda_i = \ln(R_i) / (-t_i) \quad (7.3)$$

where,

λ_i = failure rate for ith subsystem, and

t_i = mission time for the ith subsystem.

EXAMPLE 7.2: In this example where $R_i = .992$ for subsystem A, suppose that the mission time for subsystem A is 4 hours, what is the failure rate for subsystem A?

$$\text{Subsystem A failure rate} = -\ln(.992)/4 = .0080+/4 = .002+$$

7.3. Equal Allocation Process for an Exponential System

To predict the reliability for a series system the following model can be used:

$$R_{\text{sys}} = R_A \times R_B \times R_C \times \dots \times R_n$$

but when the subsystems are exponential and all have the same mission times, then the model can be written as follows:

$$R_{\text{sys}} = e^{-t(\text{system fail rate})} = e^{-t(\text{Sum of the subsystem failure rates})}$$

which implies that,

system failure rate = sum of the subsystem failure rates.

If we want to find the failure rate for each subsystem we would divide the system failure rate by the number of subsystems.

EXAMPLE 7.3: For example, suppose a system with five subsystems (each of which must operate for the entire mission time) has a reliability requirement of .96 for a 10-hour mission:

$$\begin{aligned} R(10) &= .96 = e^{-\lambda t} \Rightarrow \ln (.96) = -\lambda t = -10\lambda \text{ but,} \\ \ln (.96) &= -.04082 = -10(\lambda) \\ \Rightarrow \lambda &= .004082 \text{ failures/hour} \end{aligned}$$

Here, .004082 is the sum of the failure rates for the five subsystems.

To allocate an equal failure rate to each subsystem, .004082 failures/hour is then divided by the number of subsystems, 5.

Therefore, the allocated failure rate for each subsystem is:

$$\begin{aligned} \text{Allocated subsystem failure rate} &= .004082 \text{ failures per hour}/5 \\ &= .000816 \text{ failures/hour.} \end{aligned}$$

The process just described can be written as a single equation:

$$\lambda = \frac{-\ln(R_{\text{sys}})}{(\text{number of subsystems})(\text{mission time})} \quad (7.4)$$

Note: If the operating times for the subsystems are not the same as the mission time, the system reliability requirement can still be allocated equally to each subsystem, but the subsystem failure rates will not be equal. To compute: first divide by the number of subsystems; then divide by the operating times to yield the subsystem failure rates. This is illustrated in the following example.

EXAMPLE 7.4: Suppose a system has four subsystems and a reliability requirement of .95 for a ten hour mission. What failure rates could be allocated to the subsystems if the subsystem operating times are 1 hour, 2 hours, 5 hours and 10 hours, respectively?

Using the equation above the natural log of .95 is $-.05129+$; hence the subsystem failure rates are:

Subsys	$-\ln(R_{\text{sys}})/(\text{\#of Subsys})(t_i)$	$= \lambda_i$	$R_i = e^{-\lambda_i t_i}$
A	.05129/(4) (1)	= .0128+	.9872+
B	.05129/(4) (2)	= .0064+	.9872+
C	.05129/(4) (5)	= .0025+	.9872+
D	.05129/(4) (10)	= .0013-	.9872+

As a check, system reliability = $R_A \times R_B \times R_C \times R_D = .95$

[Note: The (4) in the second column represents the number of subsystems.]

The numbers in the last column are derived using the exponential failure model. This column was added as a check to show that in this equal allocation model each subsystem has the same reliability and that their product, i.e., $(.9872)^4$ is .95, which is the system reliability requirement.

7.4. After the Initial Allocation

The initial allocation provides you with a starting place for the failure rates. You must now determine if those failure rates are feasible. Can these failure rates actually be achieved? How expensive are they? How much time will they take to develop them? To illustrate the next step let us return to the previous example.

EXAMPLE 7.5 (Same scenario as EXAMPLE 7.4) Suppose that from past history it is known that the following failure rates are achievable for these subsystems:

Subsystem	$\lambda_{\text{allocated}}$	$\lambda_{\text{achievable}}$	Is $\lambda_{\text{allocated}}$ achievable?
A	.0128	.0010	yes, $\lambda_{\text{allocated}} > \lambda_{\text{achievable}}$
B	.0064	.0100	no, $\lambda_{\text{allocated}} < \lambda_{\text{achievable}}$
C	.0025	.0020	yes, $\lambda_{\text{allocated}} > \lambda_{\text{achievable}}$
D	.0013	.0010	yes, $\lambda_{\text{allocated}} > \lambda_{\text{achievable}}$

Upon inspection it is apparent that subsystem B has been allocated a failure rate that is not achievable. When the achievable failure rate, .0100, is inserted into row B of the table (below) and multiplied its operating time (2 hours) a reliability of .98019867 is obtained for the subsystem B, and when the system reliability is calculated we find that it is .9432+ which does not meet the system requirement.

Subsys	$-\ln(R_{\text{sys}})/(\# \text{ of Subsys})(t_i)$	=	λ_i	$R_i = (e^{-\lambda_i t_i})$
A	.05129/(4) (1)	=	.0128	.9872+
B	[t = 2]	=	.0100	.9801+
C	.05129/(4) (5)	=	.0025	.9872+
D	.05129/(4) (10)	=	.0013	.9872+
				.9432

Since this allocation does not meet the system requirement of .95 it is necessary to make up the difference by increasing the reliability in one or more of the other subsystems.

There are several ways of doing this. One method is to temporarily remove subsystem B from the calculation and use the equal allocation model to allocate the remaining reliability to the other three subsystems.

We know: that $R_B = .9801+$, that we need .95 for the system, and that R_A , R_C , and R_D must be determined. So we can state that:

$$.95 = (.9801) \times R_A \times R_C \times R_D$$

Simplifying we get:

$$\frac{.95}{.9801} = R_A \times R_C \times R_D$$

$$\text{i.e., } .96919306 = R_A \times R_C \times R_D = (e^{-\lambda_A t_A})(e^{-\lambda_C t_C})(e^{-\lambda_D t_D}) = e^{-(\lambda_A t_A + \lambda_C t_C + \lambda_D t_D)}$$

Repeating the allocation process used above, $-\ln(.96919306) = .03129$ is the expected # of failures for Super Subsystem A,C,D for a 10-hr mission.

Subsys	$-\ln(R_A R_C R_D)/(\# \text{ of subsys})(t_i)$	=	λ_i	$R_i = (e^{-\lambda_i t_i})$
A	.03129/(3) (1)	=	.0104	.9896+
B	[t = 2]	=	.0100	.9801+
C	.03129/(3) (5)	=	.0021	.9896+
D	.03129/(3) (10)	=	.0010	.9896+
				.95

Subsystem	$\lambda_{\text{allocated}}$	$\lambda_{\text{achievable}}$	Is $\lambda_{\text{allocated}}$ achievable?
A	.0104+	.0010	yes, $\lambda_{\text{allocated}} > \lambda_{\text{achievable}}$
B	.0100	.0100	yes, $\lambda_{\text{allocated}} = \lambda_{\text{achievable}}$
C	.0021-	.0020	yes, $\lambda_{\text{allocated}} > \lambda_{\text{achievable}}$
D	.0010+	.0010	yes, $\lambda_{\text{allocated}} > \lambda_{\text{achievable}}$

It should also be noted that when other factors are considered this may not be the optimal allocation. Other factors include cost, development time, etc. For example, subsystem A has an allocated failure rate of .0104+ but .0010 is achievable and there is a relatively large gap between these two values; whereas for subsystems C and D the gap is very small. It might be less costly to increase the allocated failure rate for D and decrease the allocated failure rate for A. For example, if the allocated failure rate for D is brought up to .0015 and A's allocated failure rate is dropped to .0040 we would still meet the .95 system requirement but it might be easier or less expensive to do so.

7.5. Weighted Model

Another approach for dealing with the difficulty in EXAMPLE 7.5 is to use a weighted allocation model. Letting subsystem B's failure rate be fixed at .010, and removing B from the calculations we get, as shown above:

$R_{A,C,D} = .95 / .98019867 = .9691927$ for the remaining subsystems (A, C, D).

The failure rate multiplied by mission time for the remaining three subsystems must sum to -1 times the natural log of .9691927, i.e.,

$$\lambda_A(1) + \lambda_C(5) + \lambda_D(10) = -\ln .9691927 = .03129329$$

Based on the ratios of the achievable failure rates, let's develop λ_A , λ_C , λ_D such that $\lambda_A = \lambda_D = .5\lambda_C$.

Our task is to allocate failure rates based on a weight factor that is derived from the achievable failure rates. To do this a table is set up and the following calculations are made:

1. First multiply the failure rates that are achievable for A, C, and D (Col. 1 on the table below) by the operating times (Col. 2) to get Col 3.
2. Col. 4 (Weight Factors): Sum Col. 3, then divide each value in Col. 3 by the sum of Col. 3, to get a weight factor for

each row.

3. Multiply the weight factor (Col. 4) by $(\lambda_{A, C, D})(t_{A, C, D})$ that was computed earlier (this is the mission time x failure rate that must be met by these three subsystems; it is .03129 and it represents the allocated expected number of failures for super-subsystem A,C,D). This product goes in Col. 5, it is the allocated failure rate x the mission time.
4. Col. 6 is the allocated failure rate which is found by dividing Col. 5 by "t" from Col. 2.
5. When compared to Col. 1 it is apparent that the allocated are achievable and as a check the sum of Col. 5 is .03129 which is what we need to achieve the system reliability.

Sub-system	(1) $\lambda_{Achievable}$	(2) t_i	(3) achievable expected #of failures = Col (1)x(2)	(4) Weight	(5) allocated expected # of failures = .03129 x Col (4)	(6) $\lambda_{allocated}$
A	.0010	1	.001	4.76%	.00149	.00149
C	.0020	5	.010	47.61%	.01490	.00298
D	.0010	10	.010	47.61%	.01490	.00149
			.021	99.98%	.03129	

7.6. Other Models

In addition to the equal allocation model and the weighted model, there are several other models.

One such model is the AGREE (Advisory Group on Reliability of Electronic Equipment) model. The AGREE model is based on unit complexity and importance. The model assumes that units are independent and that they are in series for reliability purposes.

Dynamic programming can also be used to allocate reliability. Information on these models can be found in the following sources:

Reliability in Engineering Design by Kapur and Lamberson.
Reliability Engineering (ARINC Research Corp.) edited by Von Alven.

For this course we will be using the equal allocation model.

7.7. Benefits of the Allocation Process

When the allocation process is a contractual item it forces the contractor to consider the feasibility of meeting a system

requirement early in the development process. If there are problem areas, they can be identified and action can be anticipated for dealing with the problems.

The allocation process can also lead to a better understanding of the system and its parts.

By going through the allocation process, more accurate failure rates can be determined for portions of the system that are subcontracted, and the failure rate needed can also be determined.

The allocation process can also lead to a more detailed consideration of the cost, weight, development time and other factors related to the acquisition of the system.

7.8. Summary

The primary purpose of this chapter was to introduce the concept of allocation (or apportionment) which is not mathematically rigorous but needs to be done to have a starting place for assigning failure rates to the elements of a system. The only model explained in-depth was the equal allocation model because it is an easy model to use and should be sufficient for initiating trade-offs.

Chapter 8

Reliability Modeling

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Symbols, Terms and Definitions

Active Redundancy	=	Parallel elements that are all operating at the same time.
AEG	=	Active Element Group
Block Diagram	=	A graphical method of illustrating how the elements of a system relate to mission success.
Parallel System	=	Same as Redundant system
Partial Redundancy	=	A parallel system with two or more elements in parallel of which two or more must operate successfully.
Passive Redundancy	=	See Standby Redundancy
R(system)	=	Reliability of a system
RADC	=	(The former) Rome Air Development Center
Redundant System	=	A system in which there are two or more elements available for use where at least one of them must operate successfully.
Sensing Device	=	A mechanical or electronic unit that indicates when the operating unit has failed.
Series System	=	A system in which each element must be successful for the system to be successful.
Standby Redundancy	=	System in which there are two or more elements available for use but only one of them is in operation at a time; also called Passive Redundancy.
Switching Device	=	A mechanical or electronic unit that switches from one element of a redundant system to another element.

8.1. Introduction

Reliability prediction is the process of determining the capability of a design and its ability to meet a specified reliability requirement.

The prediction process begins at the lowest level of a system for which data is available and works up toward the system level. It is a bottom-up approach. This is the opposite way in which a system requirement is allocated since the allocation process works from the top down.

8.2. Purpose

A prediction makes it possible to:

- determine what trade-offs need to be made,
- determine if the allocated failure rates meet the requirement,
- identify and rank problem areas,
- measure a contractor's progress,
- determine feasibility, and
- track progress during design, development, testing, and manufacturing.

8.3. Feasibility Prediction

This is an initial estimate made of a system or a part of a system; it is done to get an estimate of the reliability of a system which can then be compared to the system requirement. In the early years the feasibility prediction for electronic equipment was a function of the complexity of the system and the operational environment. The prediction was based on historical data using AEGs (Active Element Groups). For example, the MTBF for equipment with 1000 AEGs in an airborne environment ranged from about one to four hours; that same equipment in a benign (ground) environment had MTBFs that ranged from about twenty to seventy hours. Feasibility predictions such as this gave the designer a estimate of what to expect and an indication of the amount of work to be done if the reliability specification was higher than predicted.

8.4. Steps in Making a Prediction

1. Identify the reliability requirement.
2. Define the boundaries (what's included and what's not included in the system) for the prediction.
3. Develop a reliability block diagram. The block diagram shows what has to work for reliability purposes.

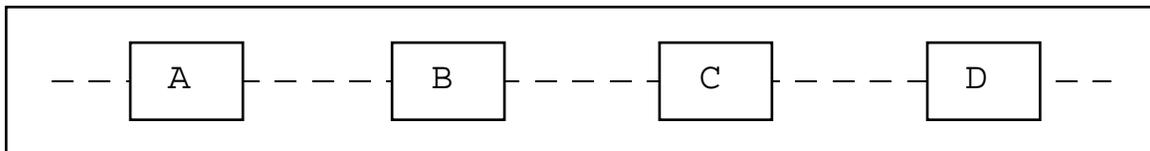
4. Develop the reliability models to the lowest identifiable function on the block diagram. This involves identifying math models for each block down to the part level. This step illustrates why it is necessary to study various statistical models since different elements of a design behave statistically in different ways.
5. Identify the stress factors for each part application.
6. Assign failure rates to each part. Failure rates for electronic parts can be found in documents such as MIL-HDBK-217; for non-electronic parts RADC has published some failure data. In either case, the data is not likely to be accurate but can be used for making comparisons.
7. Compute the reliability for each block at the part level using the math model and part failure rates.
8. Compute the reliability for each block of the block diagram using the math models developed in Step Four until the reliability for the system is obtained.

8.5 Models

The prediction models are based on the laws of probability and can be divided into series models and redundant or parallel models. The series models use the multiplication laws of probability and the redundant can use either the addition model or the multiplication model.

SERIES: A series model implies that for reliability purposes each element in the series must operate for the mission to be called a success. The functional configuration may not be in series. For example, in a car, the engine, transmission and differential are in series functionally and for reliability purposes; but the engine, cooling system, and hydraulic system which are not functionally in series would be put in series for reliability purposes because each system must operate to make a successful trip.

If a system has four elements in series a modified "block diagram" could be drawn as follows:



and the series model would be written as follows:

$$R(\text{system}) = R_A \times R_B \times R_C \times R_D$$

EXAMPLE #1: Suppose that four elements in series have the following reliabilities:

$$A = .98 \quad B = .99 \quad C = .95 \quad D = .99$$

then the reliability of the system would be:

$$R(\text{system}) = .98 \times .99 \times .95 \times .99 = .912+$$

EXAMPLE 8.2 Now let us suppose that four elements in series have failure rates and mission times as follows:

$$A = .001, 2 \quad B = .0003, 4 \quad C = .0001, 1 \quad D = .002, 3$$

and let us also suppose that the each element fails exponentially. The reliability can be calculated as follows:

$$\begin{aligned} R(\text{system}) &= e^{-[(.001)(2) + (.0003)(4) + (.0001)(1) + (.002)(3)]} \\ &= e^{-[.002 + .0012 + .0001 + .006]} \\ &= e^{-.0093} = .9907+ \end{aligned}$$

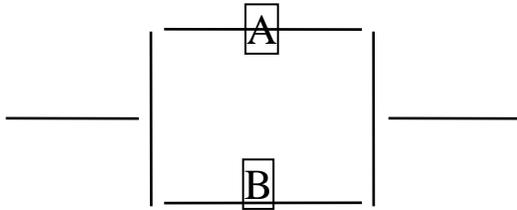
Note: This examples illustrates the way in which exponents can be added when they all have the same base, in this case "e."

This example also shows how the approximation method can be used to find the reliability, i.e., by subtracting the exponent from 1.0 (i.e., $1.0 - .0093$). This method can be used if the exponent is .1 or less. The difference in using the approximation method is small since the exact answer is .9907431.

REDUNDANT MODELS: Redundant models come in various configurations. The basic difference is active vs. passive redundancy. In active redundancy both elements are operating at all times, e.g., your eyes, ears, and nostrils are examples of active redundancy. If one of the two elements is not working you can still see, or hear, or smell but probably not as well as you could with both eyes, ears, or nostrils working. In a car head lights and tail lights are another example of active redundancy.

The purposes of using active redundancy is to increase the chance of a successful mission but it comes at the price of more weight, more spares required, more cost, more complexity, etc.

To compute the reliability of an active redundant system, two models can be used:



$$R(A) =$$

$$R(B) = .90$$

$$\begin{aligned}
 \text{Model 1: } R(\text{system}) &= R(A) + R(B) - R(A)R(B) && (8.1) \\
 &= .90 + .90 - (.90)(.90) \\
 &= 1.80 - .81 \\
 &= .99
 \end{aligned}$$

$$\text{Model 2: } R(\text{system}) = 1.0 - [(1.0 - R(A))(1.0 - R(B))] \quad (8.2)$$

but $R(A) = R(B)$, therefore, the model could be:

$$\begin{aligned}
 R(\text{system}) &= 1.0 - [1.0 - R(A)]^2 && (8.3) \\
 &= 1.0 - [1.0 - .90]^2 \\
 &= 1.0 - [.10]^2 \\
 &= 1.0 - .01 \\
 &= .99
 \end{aligned}$$

Note: If equation (8.2) is simplified algebraically the result is equation (8.1), for example:

$$\begin{aligned}
 R(\text{system}) &= 1.0 - [(1.0 - R(A))(1.0 - R(B))] \\
 &= 1.0 - [1.0 - R(B) - R(A) + R(A)R(B)] \\
 &= 1.0 - 1.0 + R(B) + R(A) - R(A)R(B) \\
 &= R(A) + R(B) - R(A)R(B)
 \end{aligned}$$

hence either model can be used. Equation (8.3) is usually easier to use because additional elements can be made redundant merely by changing the exponent.

PARTIAL REDUNDANCY: Partial redundant systems are those in which of "n" elements in parallel some number "k" of those elements must work for mission success. Laws of probability can be used to solve this problem but the Binomial model is much easier to use.

EXAMPLE 8.3 Suppose that of four elements in parallel at least two have to work for mission success. If each element has a reliability of .90, what is the probability that at least two will succeed?

To solve this problem, the probability of two (k) or more successes is calculated using the Binomial with:

$$R = .9 \text{ and } 1 - R = .1, \quad n = 4 \quad \text{and} \quad k = 2$$

$$P(k \geq 2) = P(2) + P(3) + P(4)$$

$$P(2) = {}_4C_2 (.9^2) (.1^2) = 6(.81)(.01) = .0486$$

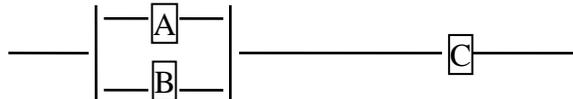
$$P(3) = {}_4C_3 (.9^3) (.1^1) = 4(.729)(.1) = .2916$$

$$P(4) = {}_4C_4 (.9^4) (.1^0) = 1(.6561)(1) = .6561$$

$$\text{-----}$$

$$.9963$$

MIXED MODELS: Mixed models are used when the block diagram has series and redundant blocks mixed together. A simple mixed model could look like this:



where C is in series with the redundant block AB.

Solving such a problem involves combining the series and redundant models using the laws of probability.

STANDBY REDUNDANCY: In a standby redundant model there are two elements in parallel but only one of them is in operation; if it should fail then the redundant element is put into operation. Two additional devices are required for standby systems, a sensing device to indicate when the operating unit has failed, and a switching device to switch the standby element into operation. We will assume the sensing device is perfectly reliable.

Case I: For two identical elements in a standby configuration with a perfect switch the reliability model for exponential failures is:

$$R(\text{system}) = e^{(-t/MTBF)} (1 + t/MTBF) \quad (8.4) \text{ because :}$$

$$R(\text{system}) = P(0 \text{ or } 1) = P(0) + P(1) = \frac{e^{-m} m^0}{0!} + \frac{e^{-m} m^1}{1!} = e^{-m} (1 + m)$$

$$\text{But } m = \lambda t = \frac{t}{MTBF}$$

$$\text{So, } R(\text{system}) = e^{-t/MTBF} \left(1 + \frac{t}{MTBF}\right) \text{ QED}$$

Case II: If the switch is not perfect, the model is:

$$R(\text{system}) = e^{(-t/MTBF)} \{1 + R(\text{switch}) (t/MTBF)\} \quad (8.5) \text{ because:}$$

$$\begin{aligned} R(\text{system}) &= P\{0 \text{ element failures or } [1 \text{ element failure and } 0 \\ &\text{switch failures}]\} \\ &= P(0 \text{ element failures}) + P(1 \text{ element failure and } 0 \text{ switch} \\ &\text{failures}) \\ &= P(0 \text{ element failures}) + P(1 \text{ element failure})P(0 \text{ switch} \\ &\text{failure} | 1 \text{ element failure}) \\ &= e^{-m} + (e^{-m} m) R_{\text{switch}} = e^{-m} (1 + m R_{\text{switch}}) = e^{-t/MTBF} \left(1 + \frac{R_{\text{switch}} t}{MTBF}\right) \text{ QED} \end{aligned}$$

where R_{switch} is independent of time and refers to a single switching operation when the first unit fails.

8.6 Summary

There are many topics that could be covered in a Reliability Course including, Open and Short Circuit Failures, Redundancy in Time Dependent Situations, Active Parallel Redundancy, Maintenance Considerations, Standby Redundancy with Unlike Elements, etc. A good reference for this material is:

Reliability Engineering, ARINC Research Corporation, Prentice Hall, Inc. 1964

Reliability: Theory and Practice, Bazovsky, Prentice Hall 1961

Another reference is MIL STD 756 which contains several models with equations for each model.

Chapter 9

WEIBULL DISTRIBUTION

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Symbols, Terms and Definitions

Beta (β)	shape parameter
Eta (η)	scale parameter
Gamma (γ) or t_0	location parameter
$f(t)$	density function
$R(t)$	reliability function
$h(t)$	hazard function

9.1. Introduction

The Weibull distribution was developed by W. Weibull, a Swedish mathematician in 1952. It was developed to analyze non-electronic systems.

The Weibull can be used to describe all three portions of the reliability bathtub curve because it has a shape parameter for each portion of the bathtub curve.

In the general case, the Weibull Distribution is a three-parameter distribution. The parameters are known as the shape parameter, the scale parameter, and the location parameter.

Beta (β) is the shape parameter. A Beta of 1.0 has an exponential shape and a constant failure rate; if Beta is greater than 1.0, the failure rate increases over time; and if Beta is less than 1.0 the failure rate decreases over time.

Eta (η) is the scale parameter, sometimes called the characteristic life, and is measured in units of time, typically hours. Eta is the time on the horizontal axis of the Weibull density function curve that divides the area under the curve into two parts, to the left of Eta the area is always .63; and to the right of Eta the area is always .37. In other words, 37% of the units will fail at an age less than Eta; and the remaining 63% of the units will fail at an age greater than Eta. Note that when Beta is exactly 1.0, the Weibull distribution becomes the exponential distribution; and the Weibull's "eta" is the same as the exponential's "theta" or mean-time-between failures.

Gamma (γ), or t_0 , is the location parameter, and is measured in units of time, typically hours. When t_0 is zero, the Weibull distribution becomes a two-parameter Weibull and is very useful for many applications. When t_0 is greater than zero, t_0 is like a guarantee period because the probability of failure is zero. Hence, t_0 represents the length of time before a failure occurs for a specific failure mode. Equipments can have a different reliability for different failure modes, for example, bearing failures vs. cracks on a bearing housing.

9.2. Application to the Reliability Bathtub Curve

The Weibull is also useful for analyzing failure data because if sufficient data is available the Weibull can often be used to determine which distribution best fits the data. **$\beta < 1$ implies "infant mortality" i.e., latent defects.**

If the analysis indicates that Beta is 1.0 then the data is most likely exponential and exponential models can be used for further analysis. **$\beta = 1$ implies "Useful Life."**

If the analysis indicates that Beta is about 2.1, the log-normal distribution may be appropriate. If Beta is about 3.44 then the data could be normally distributed and the normal distribution could be used for further analysis. **$\beta > 1$ implies "Wear-out."**

9.3. Weibull Distribution Models

Weibull Density functions:

$$\text{Two-parameter Weibull: } f(t) = (\beta/\eta) (t/\eta)^{\beta-1} \exp[-(t/\eta)^\beta] \quad (9.1)$$

$$\text{Three-parameter: } f(t) = (\beta/\eta) \{t-t_0\}/\eta\}^{\beta-1} \exp[-((t-t_0)/\eta)^\beta] \quad (9.2)$$

Weibull Reliability functions for a mission of t hours assuming the system has not been used before:

$$\text{Two-parameter Weibull: } R(t) = \exp(-(t/\eta)^\beta) \quad (9.3)$$

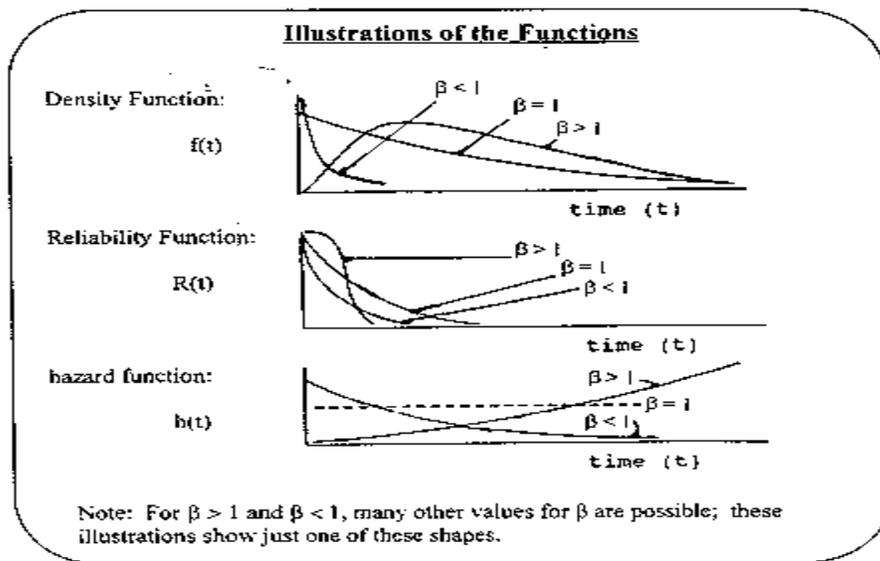
$$\text{Three-parameter Weibull: } R(t) = \exp(-((t-t_0)/\eta)^\beta) \quad (9.4)$$

Weibull Hazard functions:

$$\text{Two-parameter Weibull: } h(t) = (\beta/\eta) (t/\eta)^{\beta-1} \quad (9.5)$$

$$\text{Three-parameter Weibull: } h(t) = (\beta/\eta) [t-t_0]/\eta\}^{\beta-1} \quad (9.6)$$

9.4. Illustrations of Weibull Functions:



9.5. Graphical Method for Estimating Weibull Parameters When the Plotted Points are Co-linear.

A graphical approach will be used to estimate Beta, Eta, and t_0 . There are iterative methods available but they will not be presented in this course.

To illustrate how Eta, Beta and t_0 are estimated an example will be used.

Five failures occurred in a life test of five bearings at the following times (in hours):

1150, 2100, 700, 1600, and 2650.

Steps to estimate Beta, Eta, t_0 and the $R(800)$:

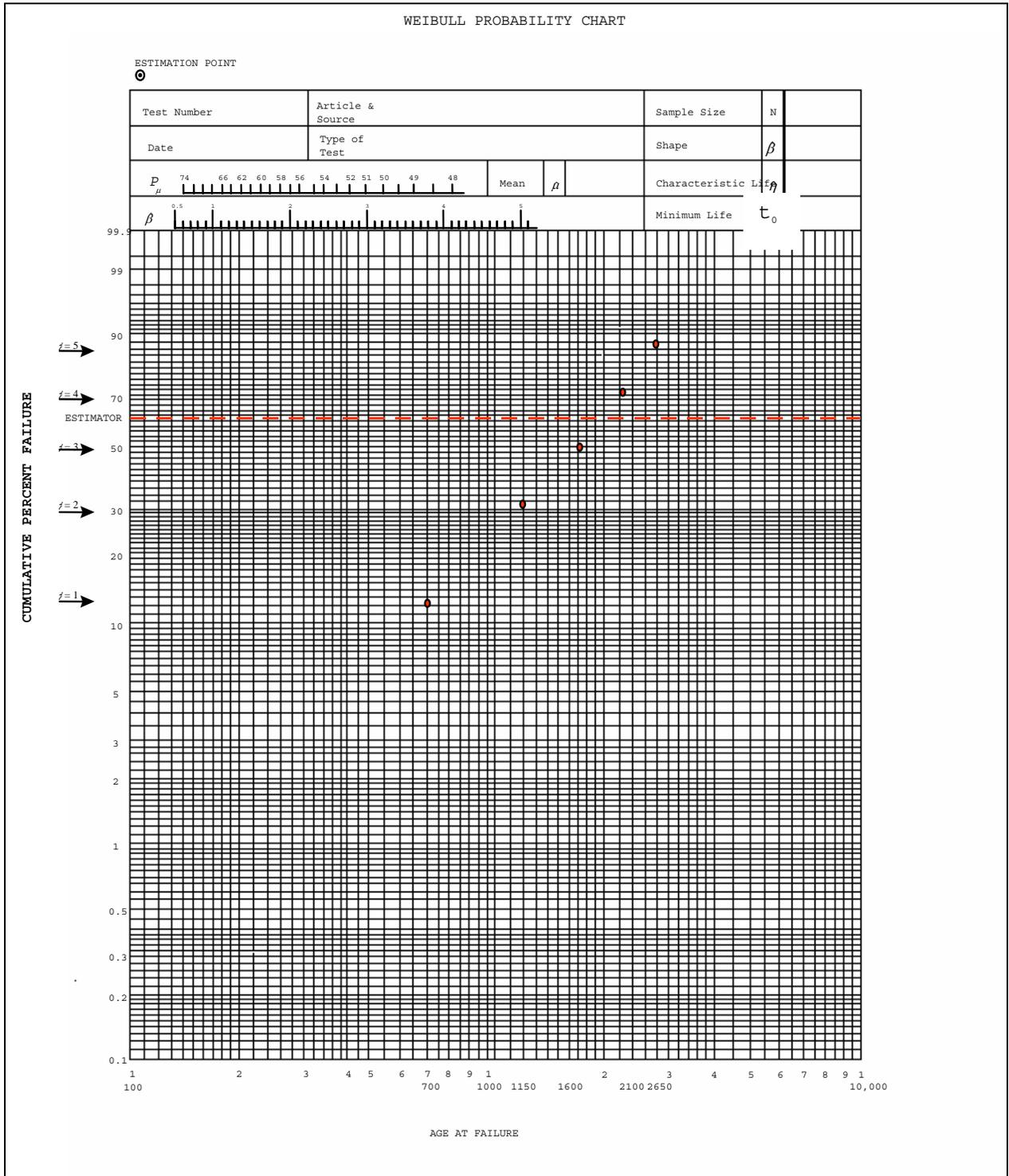
Step 1.	List the failure times in numerical order and assign a failure number (j) to each failure time beginning with j = 1 for the first failure, j = 2 for the second, etc. up to the sample size, n.	<table border="0"> <tr><td style="text-align: right;"><u>j</u></td><td style="text-align: left;"><u>Failure time</u></td></tr> <tr><td style="text-align: right;">1</td><td>700</td></tr> <tr><td style="text-align: right;">2</td><td>1150</td></tr> <tr><td style="text-align: right;">3</td><td>1600</td></tr> <tr><td style="text-align: right;">4</td><td>2100</td></tr> <tr><td style="text-align: right;">5</td><td>2650</td></tr> </table>	<u>j</u>	<u>Failure time</u>	1	700	2	1150	3	1600	4	2100	5	2650
<u>j</u>	<u>Failure time</u>													
1	700													
2	1150													
3	1600													
4	2100													
5	2650													

Step 2.	Assign a Median Rank either from the Median Rank table in Portfolio or using the equation	<table border="0"> <tr><td style="text-align: right;"><u>j</u></td><td style="text-align: left;"><u>Failure time</u></td><td style="text-align: left;"><u>Med. Rank</u></td></tr> <tr><td style="text-align: right;">1</td><td>700</td><td>12.945</td></tr> <tr><td style="text-align: right;">2</td><td>1150</td><td>31.381</td></tr> <tr><td style="text-align: right;">3</td><td>1600</td><td>50.000</td></tr> <tr><td style="text-align: right;">4</td><td>2100</td><td>68.619</td></tr> <tr><td style="text-align: right;">5</td><td>2650</td><td>87.055</td></tr> </table>	<u>j</u>	<u>Failure time</u>	<u>Med. Rank</u>	1	700	12.945	2	1150	31.381	3	1600	50.000	4	2100	68.619	5	2650	87.055
<u>j</u>	<u>Failure time</u>	<u>Med. Rank</u>																		
1	700	12.945																		
2	1150	31.381																		
3	1600	50.000																		
4	2100	68.619																		
5	2650	87.055																		
	$\text{Med. Rank} = \frac{j - .3}{n + .4}$																			

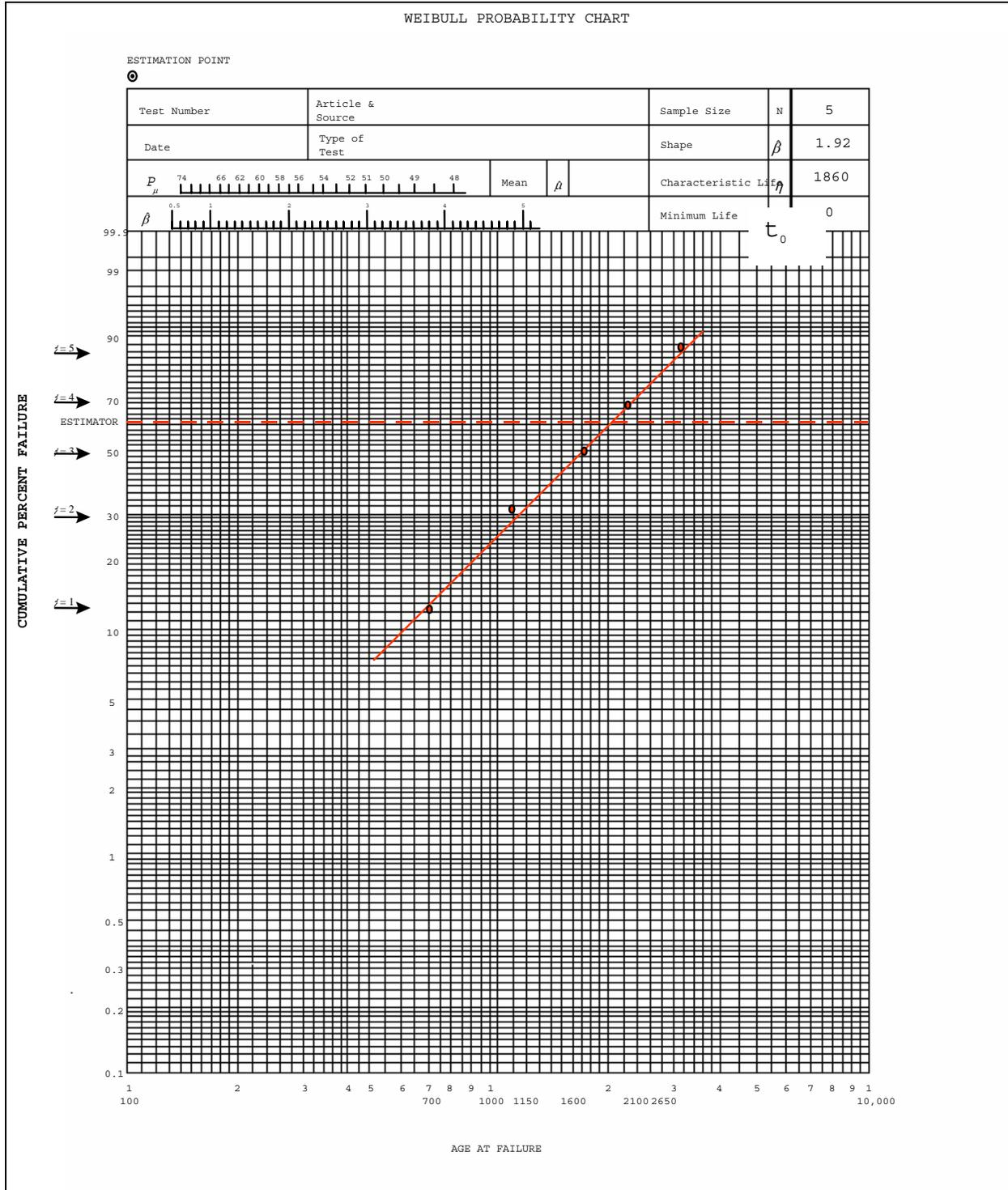
Note: Median Ranks are always based on the sample size, not the number of failures.

Step 3. On a Weibull Probability Chart, aka Weibull graph paper, change the horizontal scale to accommodate the failure time values, starting with the smallest failure time. In this example, the first "1" on the graph becomes "100." Note: The "1s" on the graph paper are always powers of "10."

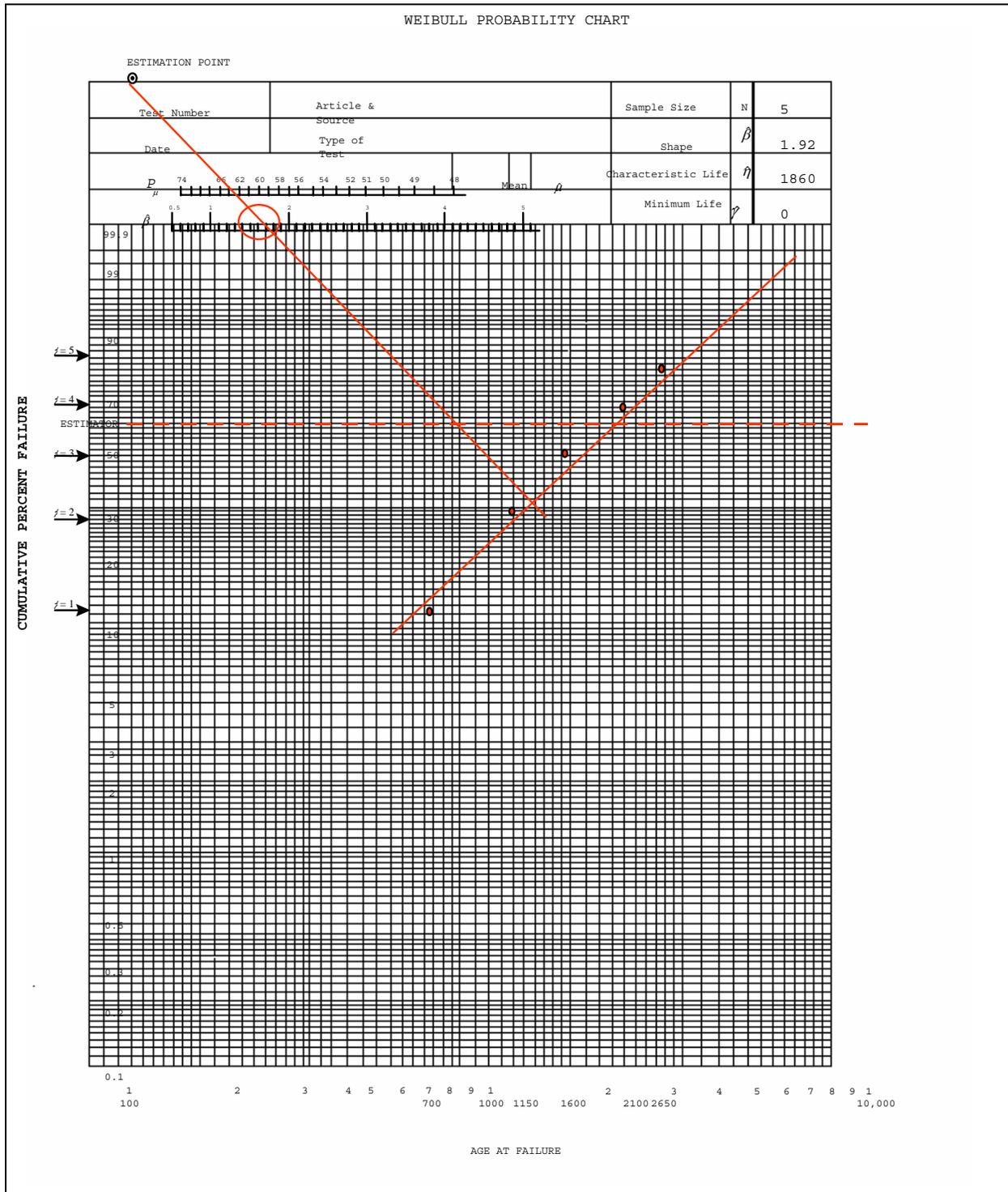
Step 4. Plot the failure times (horizontal axis) vs. the Median Ranks (vertical axis). See the graph below:



Step 5. Examine the plotted points. If they have a straight-line trend, draw the line of best fit through the points. If a straight line can be drawn, then t_0 is zero, as shown below. If not, see the example in Section 9.6.

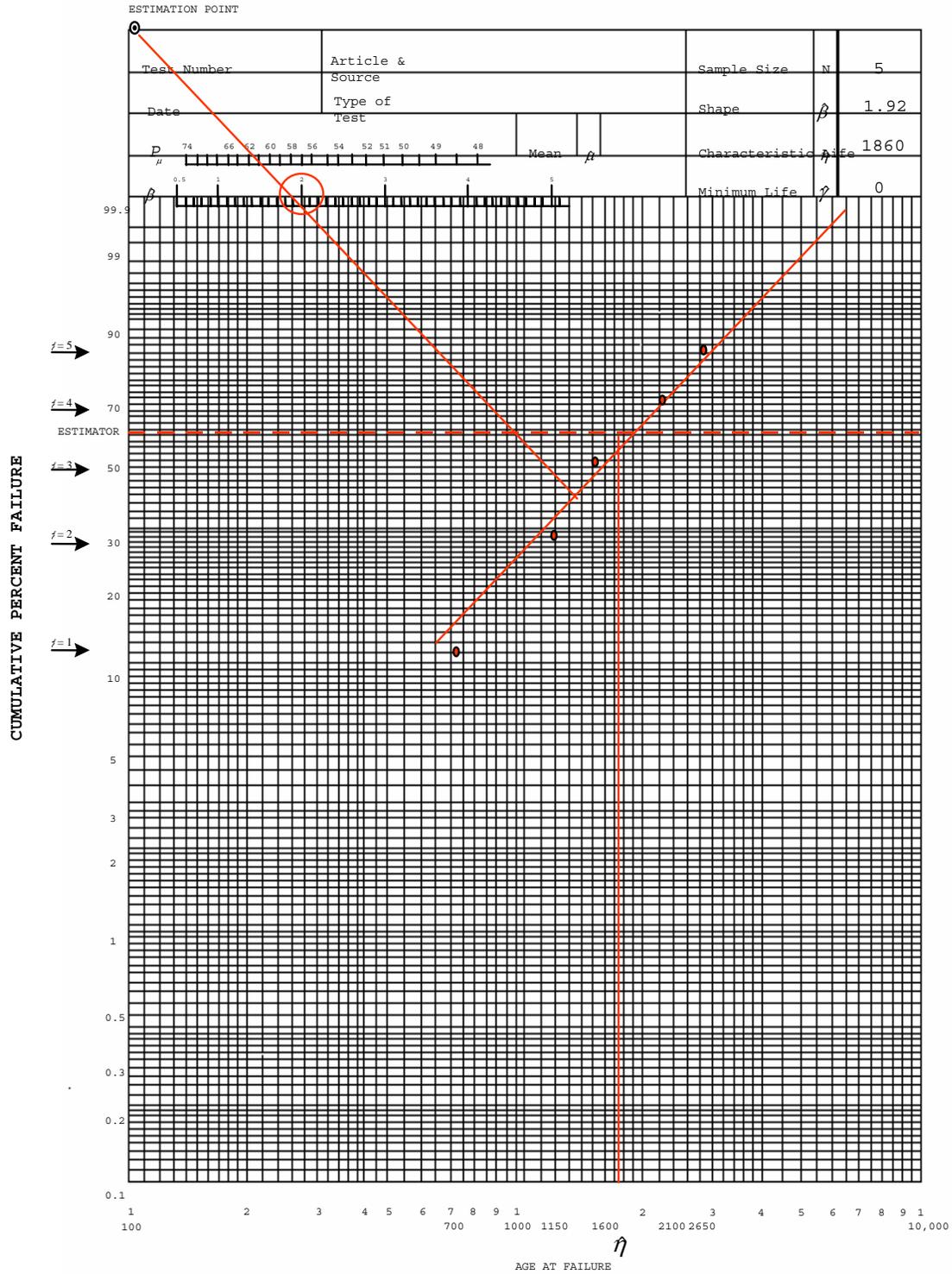


Step 6. Drop a perpendicular from the " β Estimation Point" (labeled the "Estimation Point" in the upper left hand corner of the chart below) to the line of best fit.



Step 7. **Beta and Eta:** To estimate Beta, locate the point where the perpendicular line crosses the Beta scale on the graph below. To estimate Eta, enter the Median Rank scale (vertical axis) at the 63rd percentile (dotted horizontal line). Follow the dotted line until it intersects the line of best fit. Then drop straight down to the time scale (see the chart below) to read Eta.

WEIBULL PROBABILITY CHART



Step 8. R(t): Enter the time scale at "t," and proceed straight up to the line of best fit. Go straight across to the Med. Rank scale to read the chance of failure. Subtract this number from 1.0 to find the R(t).

9.6 Graphical Method for Estimating Weibull Parameters When the Plotted Points are not Co-linear (Plot is concave downward)

When t_0 is not equal to zero and the trend of the plotted points is convex (increasing at a decreasing rate), it may be possible to straighten the line. This is done by subtracting a constant (the same constant) from each data point. See Pg. 9-10.

The constant must be a number between zero and the smallest failure time. The procedure is as follows:

Step 1. Begin by subtracting half the smallest failure time from each failure time.

Step 2. Re-plot the data.

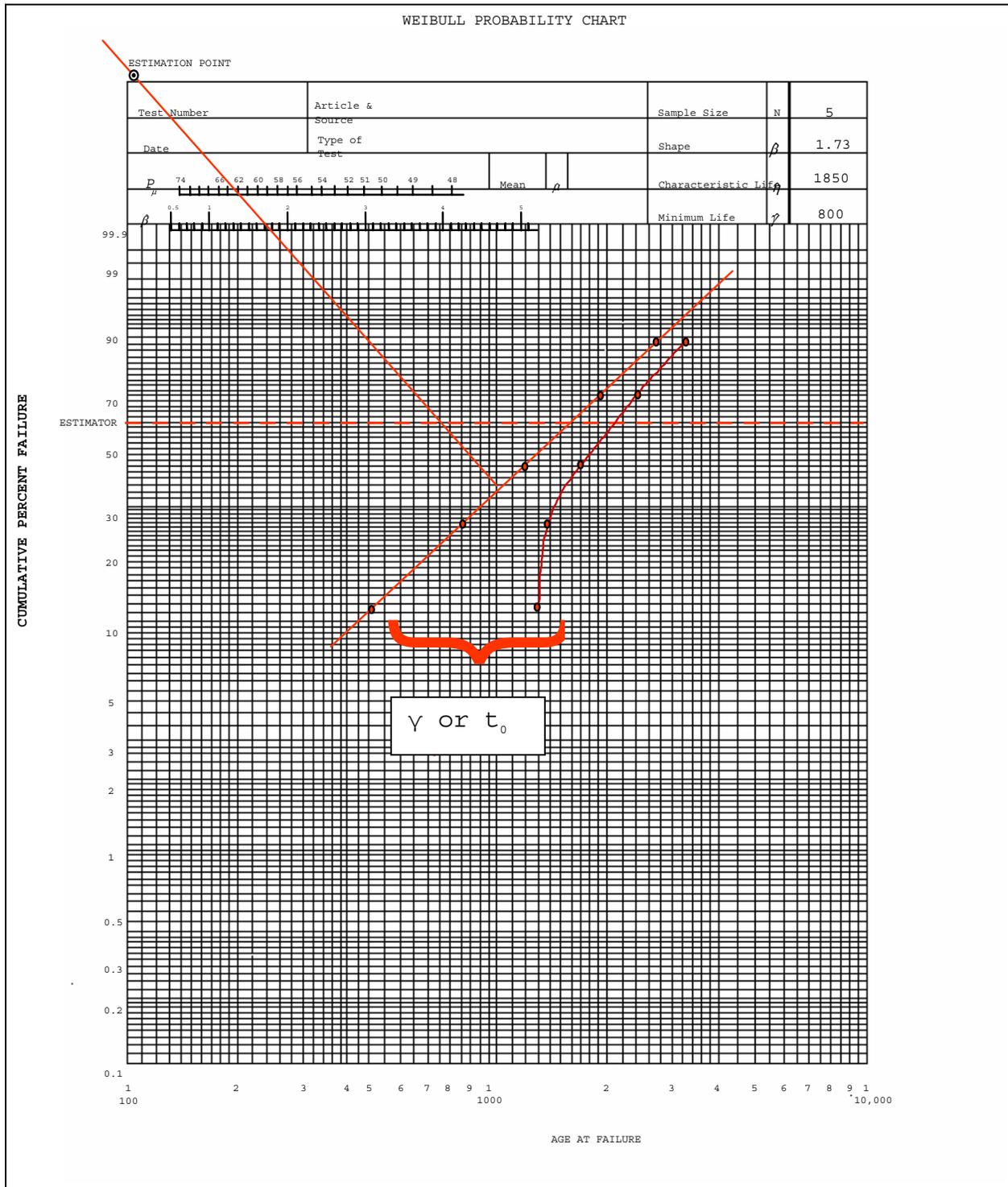
Step 3. Check for a straight-line trend. If the trend is now straight, the amount subtracted is t_0 .

Step 4.

A. If the trend is still convex, subtract a larger number, re-plot and re-check the trend to see if it is now straight. If not, continue subtracting and plotting until the plotted points have a straight-line trend. Remember, you cannot subtract a number larger than the smallest failure time.

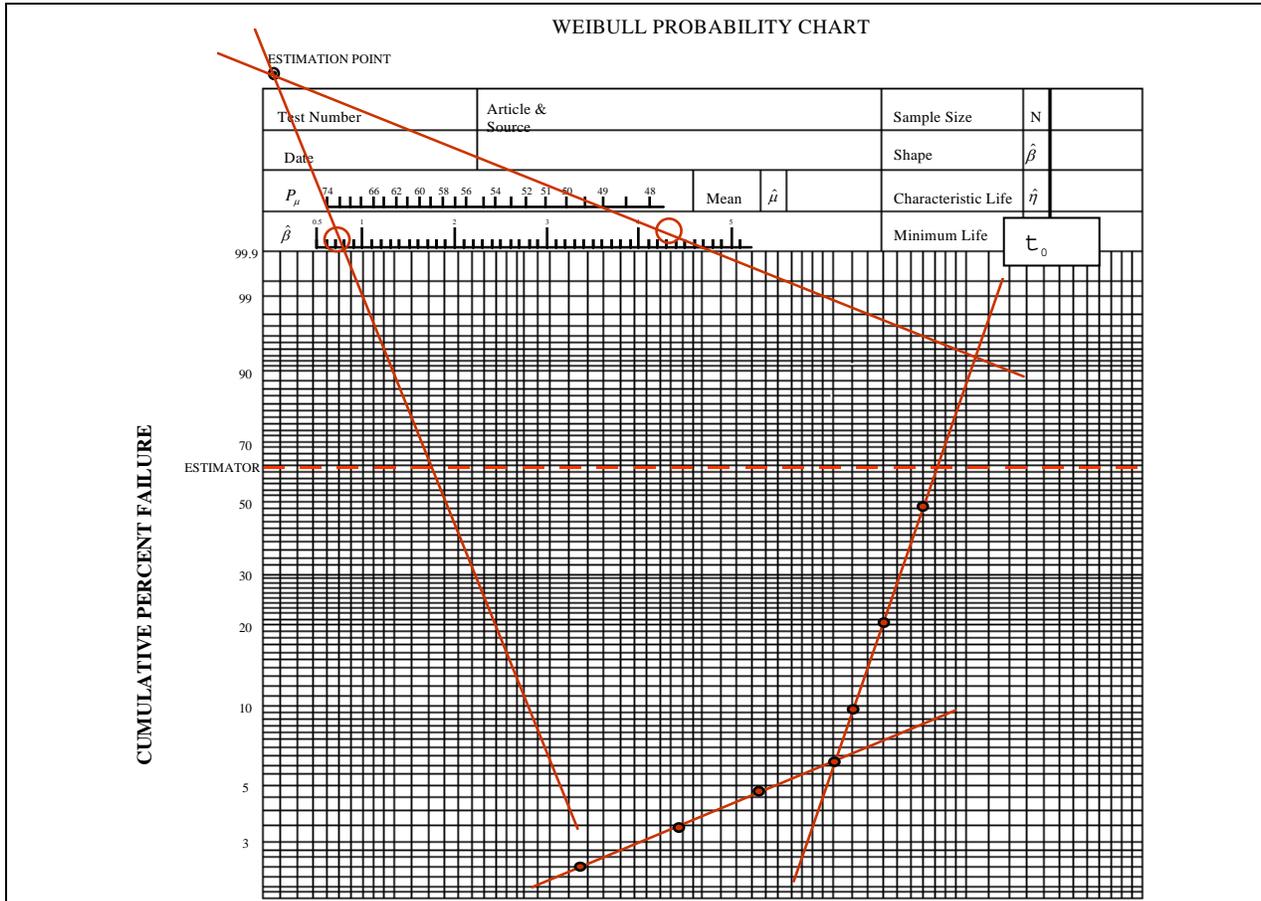
B. If at any time during this process the trend curves in the opposite direction (concave), the amount subtracted is too large; try a smaller value, re-plot and re-check.

Step 5. Continue making adjustments until a straight-line trend results, as shown in the graph below. Note: to determine the $R(t)$, enter the time scale at $(t-t_0)$. If it is impossible to get a straight line, then the data may not be well-modeled by a Weibull.



9.7. Other Applications (When the Plot is Concave Upward)

If the original trend is concave (increasing at an increasing rate), it is possible that more than one Weibull can be fitted to the data. See graph below:



In this example, the intersection point of the two straight lines could represent the age at which the failure rate changed from decreasing to a constant, or from a constant to increasing, depending on the values of Beta. It is also possible that two failure modes with different Beta values have been plotted. To check this possibility, identify the failure modes for the data; if they are different, plot them separately. Then analyze the revised plots.

9.8. Confidence Intervals

It is possible to make confidence statements for the Weibull from the graph. To do this:

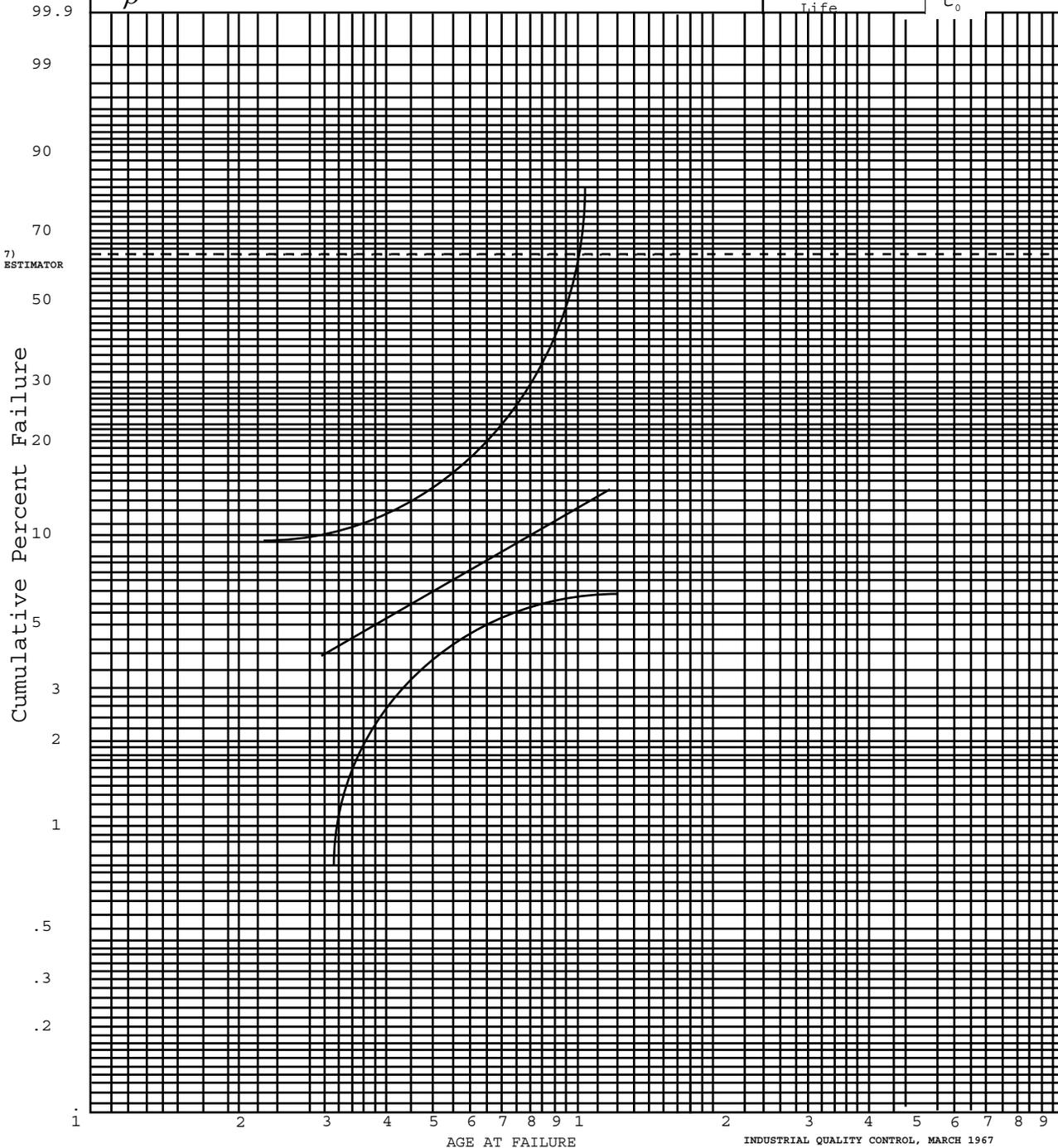
1. Find the 5% and 95% tables and read the ranks for those tables for each failure time.
2. Plot the time vs. ranks.
3. Connect the plotted points.
4. The result is an upper and lower confidence band for the data (See the graph that follows).

ESTIMATION POINT

WEIBULL PROBABILITY CHART



Test Number	Article & Source	Sample Size	N
Date	Type of Test	Shape	$\hat{\beta}$
P_{μ}	Mean	Characteristic Life	$\hat{\eta}$
$\hat{\beta}$	Minimum Life	t_0	



INDUSTRIAL QUALITY CONTROL, MARCH 1967

9.9. Reliability of a Used System: Conditional Probability

If the system has been used before, the conditional probability model should be applied to compute reliability for a mission of t hours. The equation to be used is as follows:

$$R(t) = \frac{R(T_1)}{R(T)} \quad \text{i.e., } P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \quad (9.7)$$

Where,

T is the time when the mission starts,

T_1 is the time when the mission is completed, and

$T_1 - T = t =$ the mission duration.

To calculate mission reliability the Weibull model is used to compute $R(T_1)$ and $R(T)$.

For example, the reliability for a ten-hour mission on a system that already has 100 hours of use is desired. This means that the mission starts at the 100th hour and ends at the 110th hour.

The system is Weibull distributed with:

$$\begin{aligned} \text{Beta} &= 1.5, \\ \text{Eta} &= 300 \text{ and} \\ t_0 &= 0 \end{aligned}$$

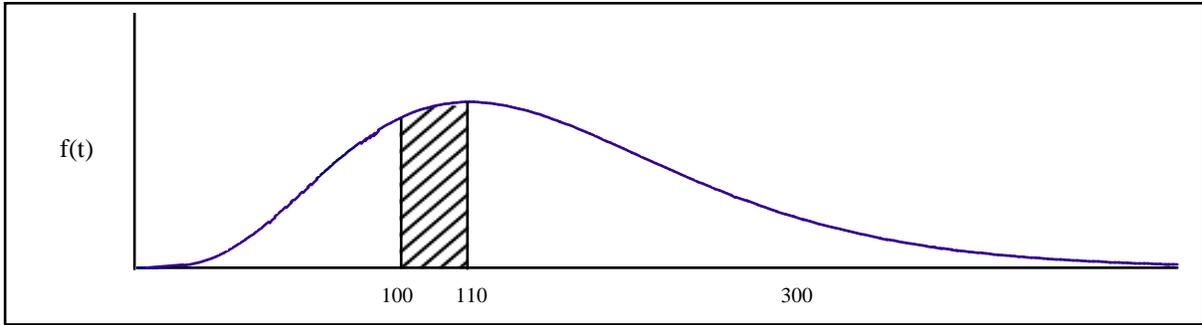
Therefore,

With $T = 100$ and $T_1 = 110$ hours

$$R([100-110] | [0-100]) = \frac{R(0-110)}{R(0-100)}$$

Where $R(0-110)$ is the probability a new unit will survive 110 hours; and $R(0-100)$ is the probability a new unit will survive 100 hours.

$$R([100-110] | [0-100]) = \frac{\exp(-110/300)^{1.5}}{\exp(-100/300)^{1.5}} = \frac{.801}{.825} = .97$$



9.10. Hazard Rate

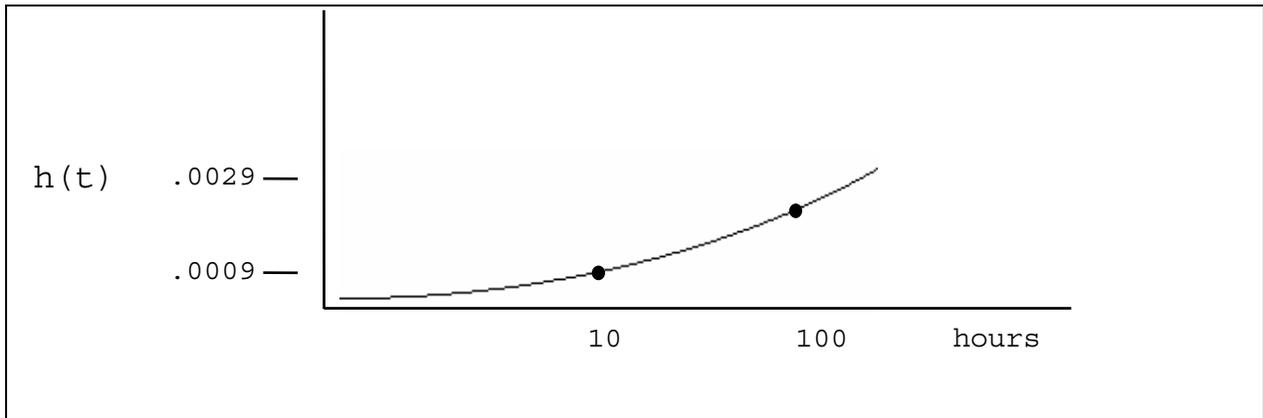
As stated earlier, if Beta is less than one the hazard rate decreases, if Beta is larger than one the hazard rate increases.

If a system's failures are Weibull-distributed with:

$$\begin{aligned} \text{Beta} &= 1.5, \\ \text{Eta} &= 300 \text{ and} \\ t_0 &= 0 \end{aligned}$$

We can compute the hazard rate at 10 and 100 hours as follows:

$$\begin{aligned} h(10) &= (1.5)/(300) \times (10/300)^{(1.5 - 1)} \\ &= (.005) \times (.183) = .0009+ \\ h(100) &= (1.5)/(300) \times (100/300)^{(1.5 - 1)} \\ &= (.005) \times (.577) = .0029+ \end{aligned}$$



As you can see the failure rate is about three times higher at 100 hours than it was at 10 hours.

9.11 Summary

The Weibull is a versatile distribution and can be useful to anyone involved in analyzing test or field data. It is important that the database be large enough so that an accurate analysis is possible.

NOTE: This ARTICLE APPEARED IN "Introduction To Weibull Analysis" by the Statistical Engineering Group at Pratt & Whitney Aircraft. Revised March 15, 1975.

APPENDIX F

ANALYSIS OF TWO FAILURE DISTRIBUTIONS

Plotting failure data which contain two or more different modes of failures (e.g., ball failures or inner bearing race failures) will normally not result in a straight or smooth curved ($t_0 \neq 0$) Weibull line. From Figure 16 in the text, we saw that the data described neither a straight or smooth curved line. Yet with a special analysis, we were able to separate the failure distributions. Consider the following data from a bearing life study program as a practical example of (1) data we might expect some day and (2) a method of analyzing this data.

Order Number	Hours-to-Failure	Failure Mode
1	7.8	Ball
2	20	Inner Race
3	22	Ball
4	50	Ball
5	90	Inner Race
6	102	Ball
7	120	Inner Race
8	140	Ball
9	224	Ball
10	300	Inner Race

A Weibull plot of these ten data points is shown in Figure F-1.

LIFE CHARACTERISTICS (WEIBULL METHOD)

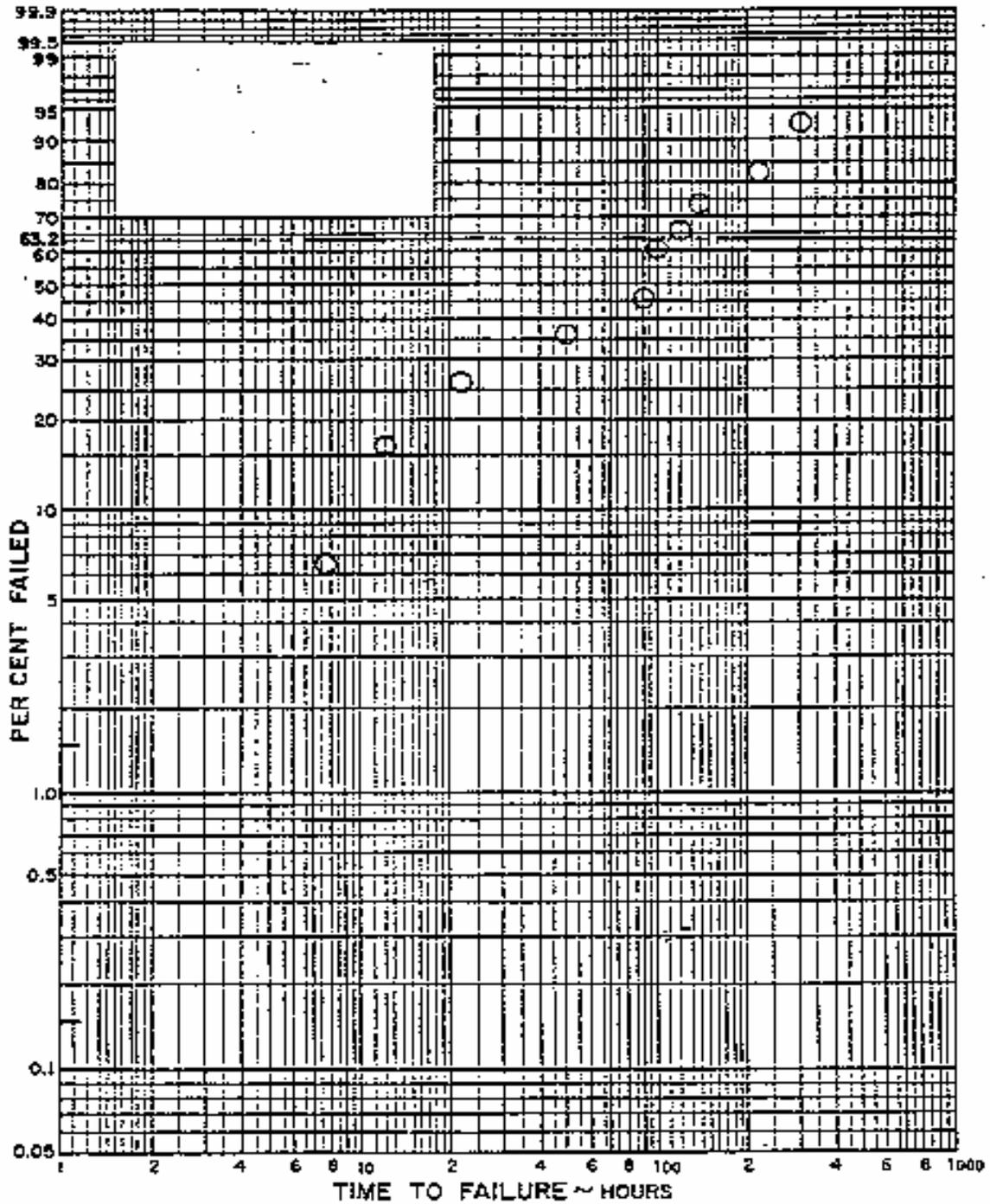


Figure F-1 Weibull Plot of All Data Points.

Obviously, the data do not describe a straight line on Weibull probability paper. A special analysis is necessary to separate the two failure modes. The correct way of handling these data is to:

1. Identify the failure mode for each bearing, by technical inspection.
2. Separate the data by failure modes (ball or inner race)
3. Replot the data

When replotting the data for the ball failures, the inner race failures are considered suspension. On the other hand, the ball failures are considered suspensions when plotting inner race failures. (See Appendix D, "Analysis of Test Data which Include Suspended Data").

Ball Failures			
<u>Order Number</u>	<u>Hours-to-Failure</u>	<u>Rank Order Number</u>	<u>New Median Rank</u>
1	7.8	1	6.697
2	22	2.111	17.295
3	50	3.222	28.000
4	102	4.518	40.513
5	140	6.139	56.174
6	224	7.760	71.826

Six failures were due to the ball failing. The other failures were considered suspension. The failures were ranked in ascending order and a new rank order number and median rank were calculated for each failure. Figure F-2 describes the distribution of ball failures.

LIFE CHARACTERISTICS (WEIBULL METHOD)

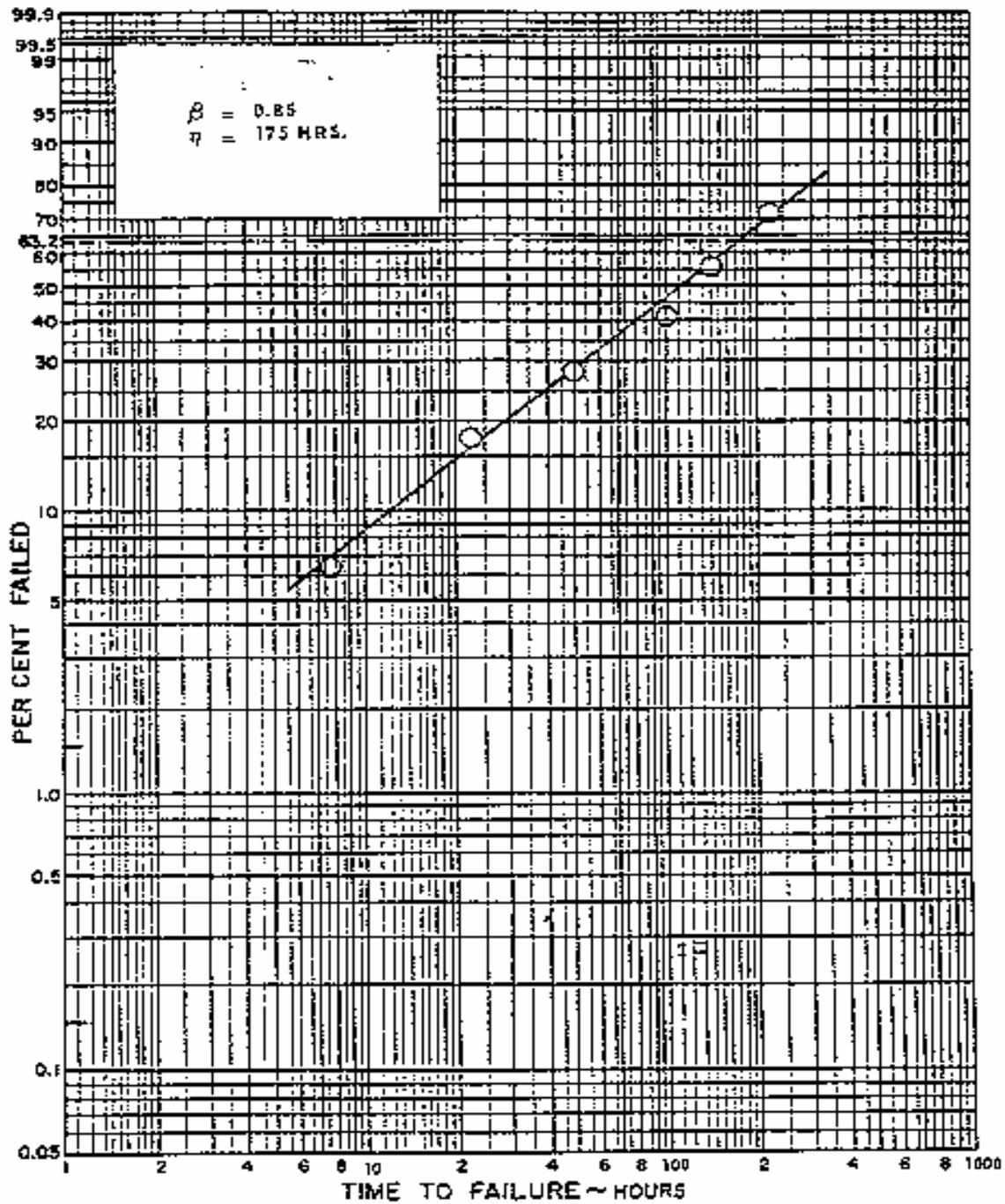


Figure F-2 Weibull Plot of Sail Failures.

<u>Inner Race Failures</u>			
<u>Order Number</u>	<u>Hours-to-Failure</u>	<u>Rank Order Number</u>	<u>New Median Rank</u>
1	20	1.100	7.650
2	90	2.514	30.773
3	120	4.211	37.548
4	300	7.605	70.329

The remaining failures were attributed to the inner race failing. The ball failures were considered suspensions. Again, new rank order numbers and median ranks were calculated and the data were replotted in Figure F-3

LIFE CHARACTERISTICS (WEIBULL METHOD)

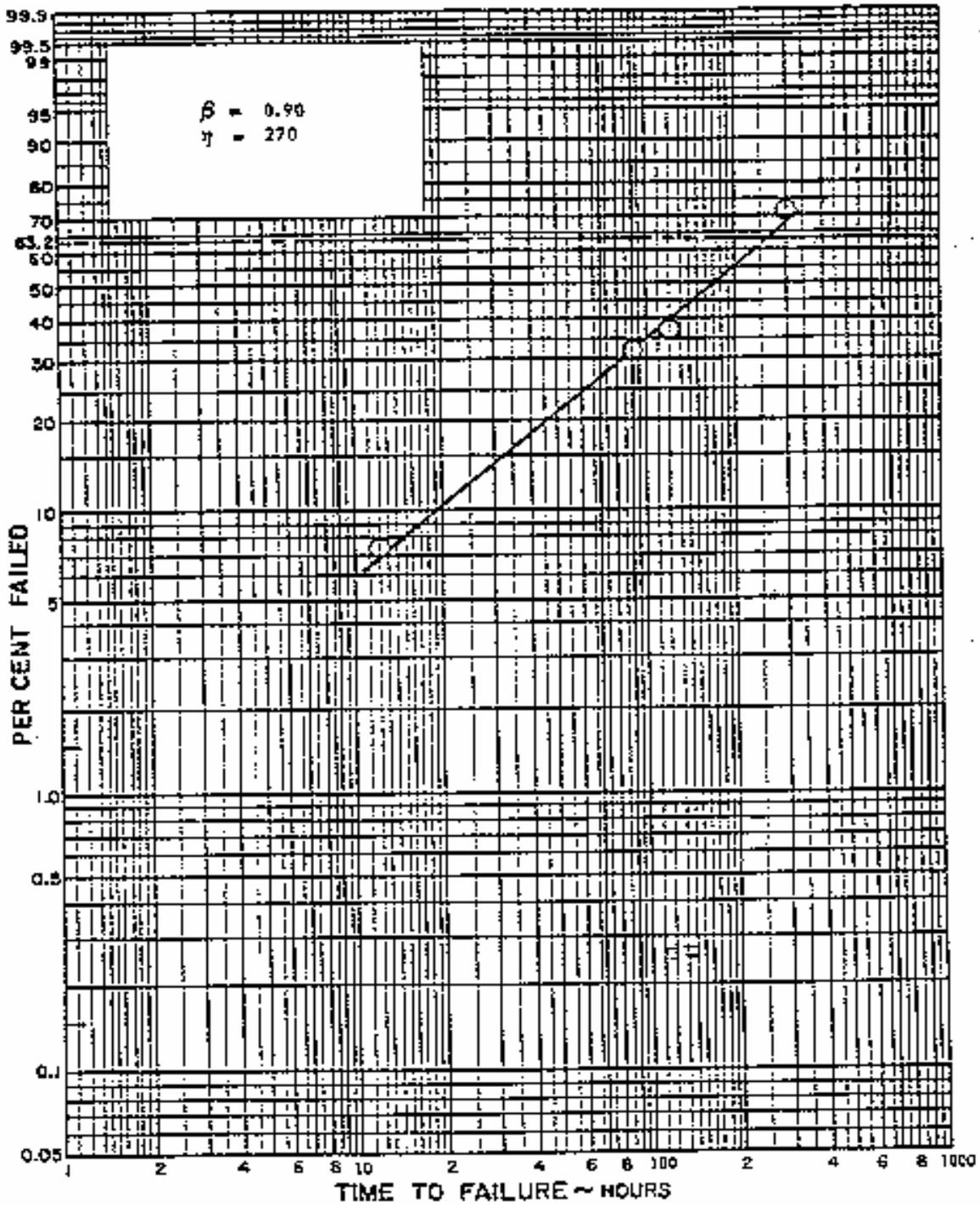


Figure F-3 Weibull Plot of Inner Race Failures.

Chapter 10

Reliability Growth; Test, Analyze and Fix

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10.1. Background

The new graduate engineer in the office has been given his first assignment - to design a new, "more reliable" mousetrap - in four weeks! Enthusiastic, he spends week one assessing the behavior and physical characteristics of mice. He evaluates available materials, and the strengths and weaknesses of existing designs. He formulates and then refines his new design in the CAD system during the second week. Assured that he has the perfect design, he transmits his "drawings" to the prototype shop and to the production department. By the end of the third week, the prototypes are completed - in exact accordance with his design specifications. Next week, the units are set up for functional testing in the lab. All goes well for several days until . . . one morning the mouse escapes! The trap has failed! And the boss wants to begin production next week!

The above situation illustrates one reality of the design process - that no matter how good the designer or how helpful the CAD system, NO design can be expected to be perfect coming off the "drawing board." The functional requirements may push the state-of-the-art. Actual environmental conditions may differ from those expected. Wrong parts or materials may be selected. And analyses and/or supporting data may be in error. Thus, the initial paper design will only be partially complete, the degree of completeness depending upon the skill of the engineers, resources available and other factors. To complete the design, the actual hardware/software must be operationally tested, remaining design imperfections must be identified, and permanent corrective action must be implemented and verified. This process of refining the design is sometimes called Reliability Growth.

10.2. Reliability Growth

Reliability growth is defined in MIL-HDBK-189 (Ref 1) as:

"The positive improvement in a reliability parameter over a period of time due to changes in product design or the manufacturing process."

This definition identifies three aspects of reliability growth:

- It can be measured,
- It takes time, and
- It requires change in the product and/or processes.

10.3. Measuring Reliability Growth

Reliability is generally quantified in one of three ways:

1. As a probability (e.g., the ratio of successful missions to total missions)
2. As a rate (e.g., the number of "failures," removals, maintenance actions, etc. per unit time) or
3. As an average interval between events (e.g., Mean Time Between Failure (MTBF), Maintenance Actions (MTBMA), etc).

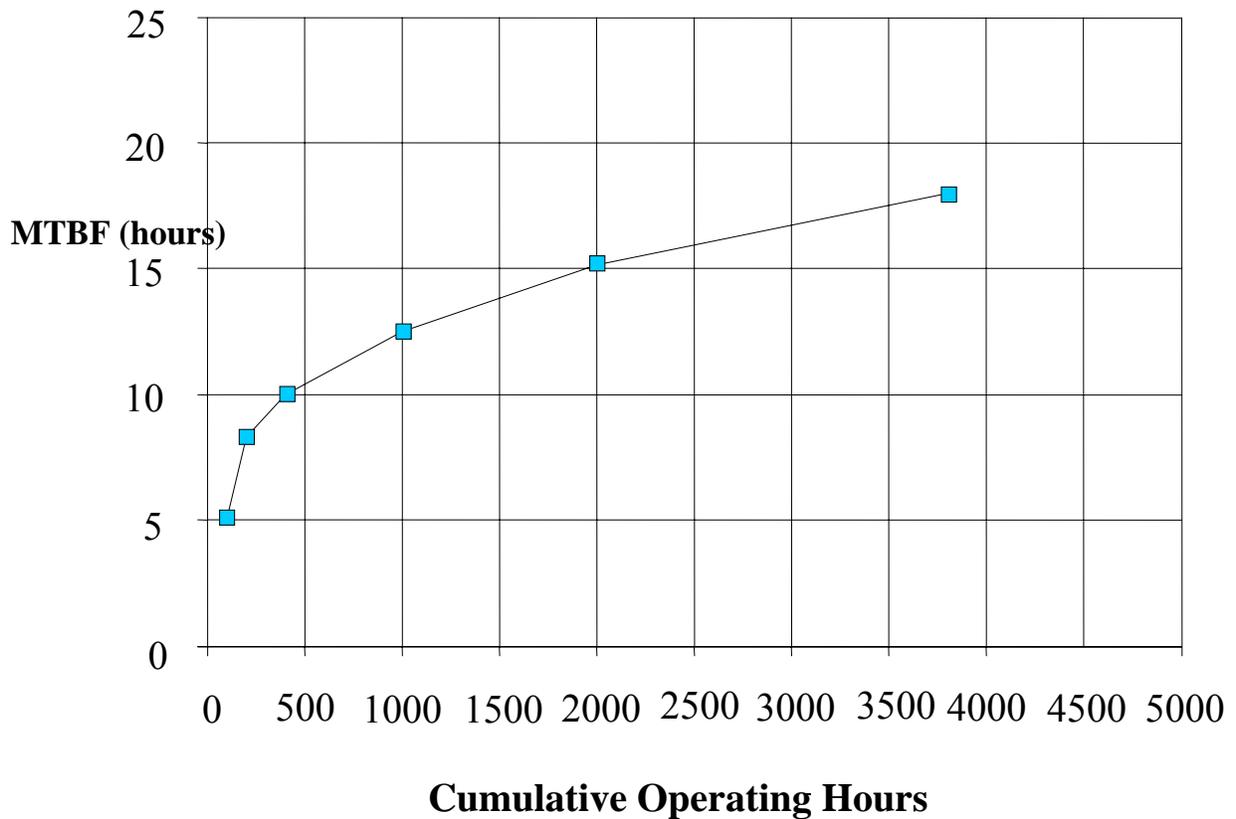
Any of the above measurements can be used to assess the current status of reliability and to track the improvement as a result of applying the TAAF process. In this chapter we will use the third type of measurement to discuss tools to help plan for and track reliability growth.

To illustrate, the following data were recorded for six consecutive periods of system operation.

<u>Period</u>	<u>Operating Hours</u>		Failures in Each <u>Period</u>	MTBF in Each <u>Period</u>
	<u>Period</u>	<u>Cum</u>		
1	100	100	20	5.0
2	100	200	12	8.3
3	200	400	20	10.0
4	600	1000	48	12.5
5	1000	2000	66	15.2
6	1800	3800	100	18.0

The data indicate that the reliability (i.e., MTBF) improved from 5 hours to 18 hours. Figure 1 shows this reliability growth graphically.

Figure 1
Reliability Growth Curve



10.4. Reliability Growth - Time

"As a generalization, it might be said that the built-in reliability is proportional to the time allowed the supplier to design, debug, produce a pilot run, and incorporate changes or improvements in production as a result of feedback from field experience . . . Time is the mortar which binds this whole structure of reliability together . . . Reliability cannot be achieved on a crash basis . . ." (Ref 2)

As the quotation indicates, sufficient time is required to achieve the desired reliability. Furthermore, that time may be during the design/debug phase, the production phase, or the fielded phase. In effect, reliability growth can take place anytime during the acquisition cycle where hardware/software is available for testing. Growth processes can also be applied at a variety of levels of assembly - from a "breadboard card" in the lab as well as to a major subsystem in a newly deployed aircraft. In this chapter we will encourage the use of reliability growth techniques during the engineering and manufacturing development (EMD) phase of the acquisition cycle. However, these techniques can be applied before and/or after EMD.

10.5. Reliability Growth - Change

Changing materials or a part in the design is one way to correct a problem. Example: The F-111A radar subsystem was experiencing frequent failures of expensive traveling wave tubes. Five of these \$66,000 tubes were replaced at one base in a single month period! Through engineering analysis, the fault was traced to an original design error. The value of a single resistor was not correct for the specific application. Resistors with the correct design value were installed and the cost of supporting the system was reduced significantly.

Other problems may be related to the quality of the manufacturing process. A two-million dollar missile was lost during a flight test due to a tiny fragment of foreign material lodged in a coil of very fine wire. The fragment eventually cut through the insulation of the wire and caused an electrical short! No change in the design was required, but improved quality was considered.

Change is the key to effective reliability growth. Just repairing a broken unit does not achieve reliability growth.

10.6. Reliability Growth and the TAAF Process

There are three steps to achieving change that results in reliability growth; they are frequently called "test, analyze, and fix," or simply TAAF.

TEST! ANALYZE! and FIX!

➤ Test - Stimulate Failures

Test means to operate the equipment so that deficiencies can be observed. Both hardware and software are designed to perform specific functions under expected operational environments. For example, a farm tractor is expected to be able to develop enough traction and power to plow hard ground. The engine and other components must operate in dirty/dusty conditions continuously for many hours without servicing or repair. Deficiencies in the design of a farm tractor will not appear if the tractor is tested by just placing it in a barn with the engine idling. The tractor must be placed under STRESS to see if the design will perform necessary functions reliably.

It is sometimes impractical to place test articles in the actual environment. Space systems are good examples - it would be extremely difficult and costly to try to debug a satellite in space! An alternative to taking the test articles to the actual environment is to bring a simulated environment to the test article. For example, the Air Force has wind tunnels to simulate high altitudes and supersonic air flow for testing aircraft and missile engines. For airborne electronic systems, testing is frequently performed in relatively small chambers on the ground. There, environments such as temperature, vibration, humidity, power quality and on/off cycling can be varied to simulate the operational environment. Section 10-12 has additional information on environments.

➤ Analyze - Root Causes

Analyze means to determine the root cause of the problem. For example, just finding and replacing a failed integrated circuit (IC) on a card is not sufficient. The IC must be carefully studied and disassembled if necessary to find the root cause. For example, the root cause of the failure of a critical IC in a space shuttle computer was human spittle. Sealed into the package at the time of manufacture, the spittle eventually corroded the minute features on the IC and caused it to fail. Also possible is the fact that the IC is really OK and the real cause of the malfunction is elsewhere, such as a bad solder joint.

➤ Fix - Permanent Corrective Action

Fix involves developing a new design or manufacturing process to permanently improve the product. This means more than just incorporating a change. It also involves retesting the equipment to verify that the fixes are effective, permanent, and do not induce additional new failure modes. One reliability growth expert claimed that approximately 30% of all corrective actions were ineffective. (Ref 3)

10.7. Informal TAAF

As indicated above, the process of testing, analyzing, and fixing takes time and other resources such as test assets, test chambers or other facilities, manpower, etc. Some TAAF may occur spontaneously as the results of malfunctions that occur during normal development testing. Corrective actions frequently result from such "informal" TAAF. But experience has shown that the best way to thoroughly "debug" a system is to plan for a formal, structured period of time to focus on this TAAF process.

10.8. Formal TAAF

Formal TAAF, sometimes called a Reliability Development/Growth Test (RD/GT), is documented in a plan, generally a part of the overall contractual R&M Program Plan (See section 10-18 for additional information). A TAAF process should continue until a predetermined number of hours have been accumulated, or preferably until the system has achieved a desired level of reliability. Questions: How many hours should one plan for? How many test assets are required? What level of reliability can be achieved in a certain period of time?

10.9. The Duane Reliability Growth Model

As we saw in figure 1, the MTBF continued to improve as additional test time was accumulated. If there was a way to quantify the relationship shown in that figure, the above questions might be answered.

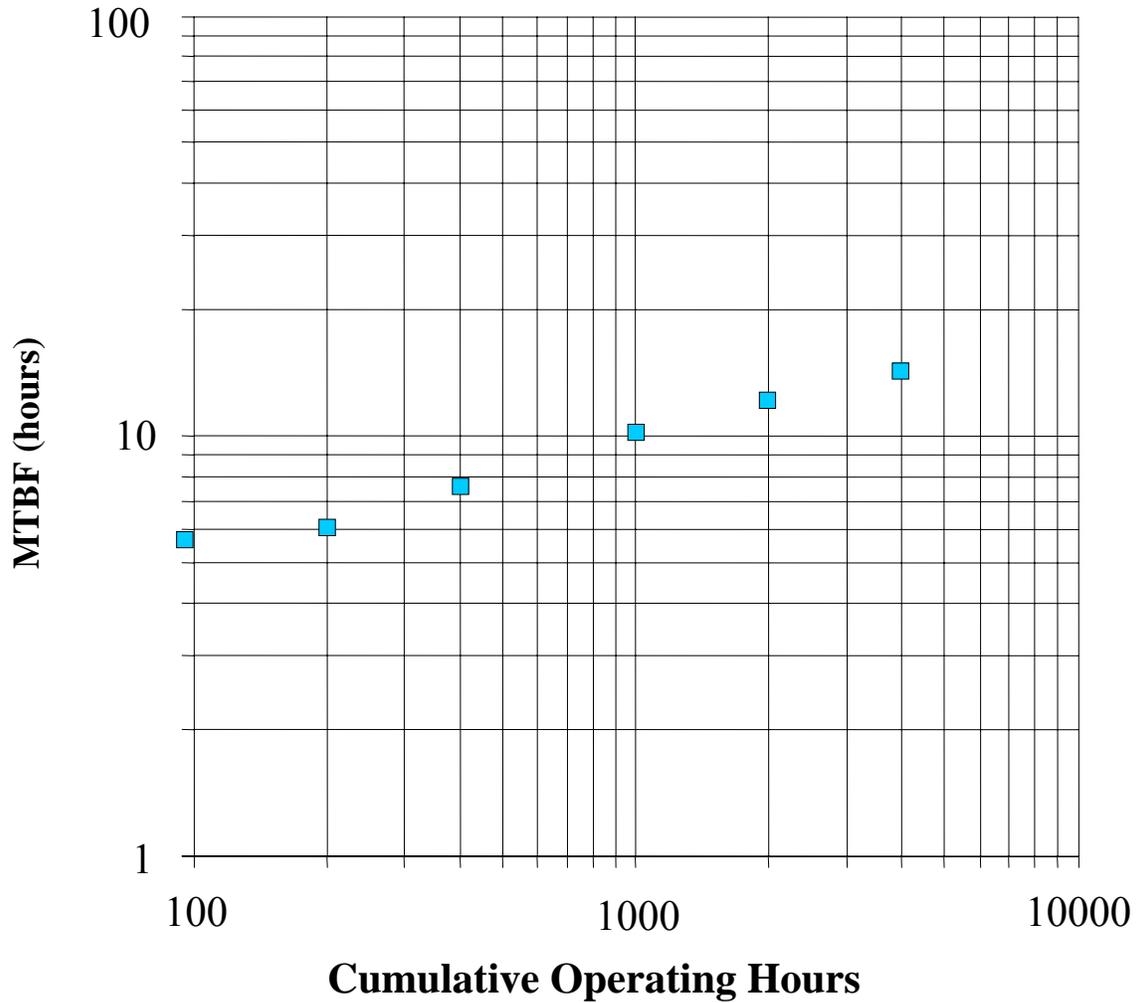
In 1962, a General Electric employee, J. T. Duane developed a simple graphical and mathematical model of the reliability growth process (Ref 4). He collected and analyzed large quantities of time and failure data on several systems undergoing test. He discovered that if he plotted cumulative values of operating test time and failure rate on logarithmic (log-log) paper, that the data points approximated a straight line. We can use our original data to illustrate Duane's observation. Note: Duane's original data was expressed in terms of failure rates, but for most applications, we prefer to use Mean Time Between _____ values such as MTBF, MTBR, and MTBM.

Step 1: Calculate the Cumulative MTBF values as follows:

Period	Operating Hours For Period	Operating Hours Cumulative	Failures in Each Period	MTBF in Each Period	Cumulative failures	Cumulative MTBF
1	100	100	20	5.0	20	5.0
2	100	200	12	6.3	32	6.3
3	200	400	20	7.7	52	7.7
4	600	1000	48	10.0	100	10.0
5	1000	2000	66	12.0	166	12.0
6	1800	3800	100	14.3	266	14.3

Step 2: Plot cumulative MTBF values vs. cumulative operating hours on log-log paper (see graph below). As you can see, the data points are approximately in a straight line.

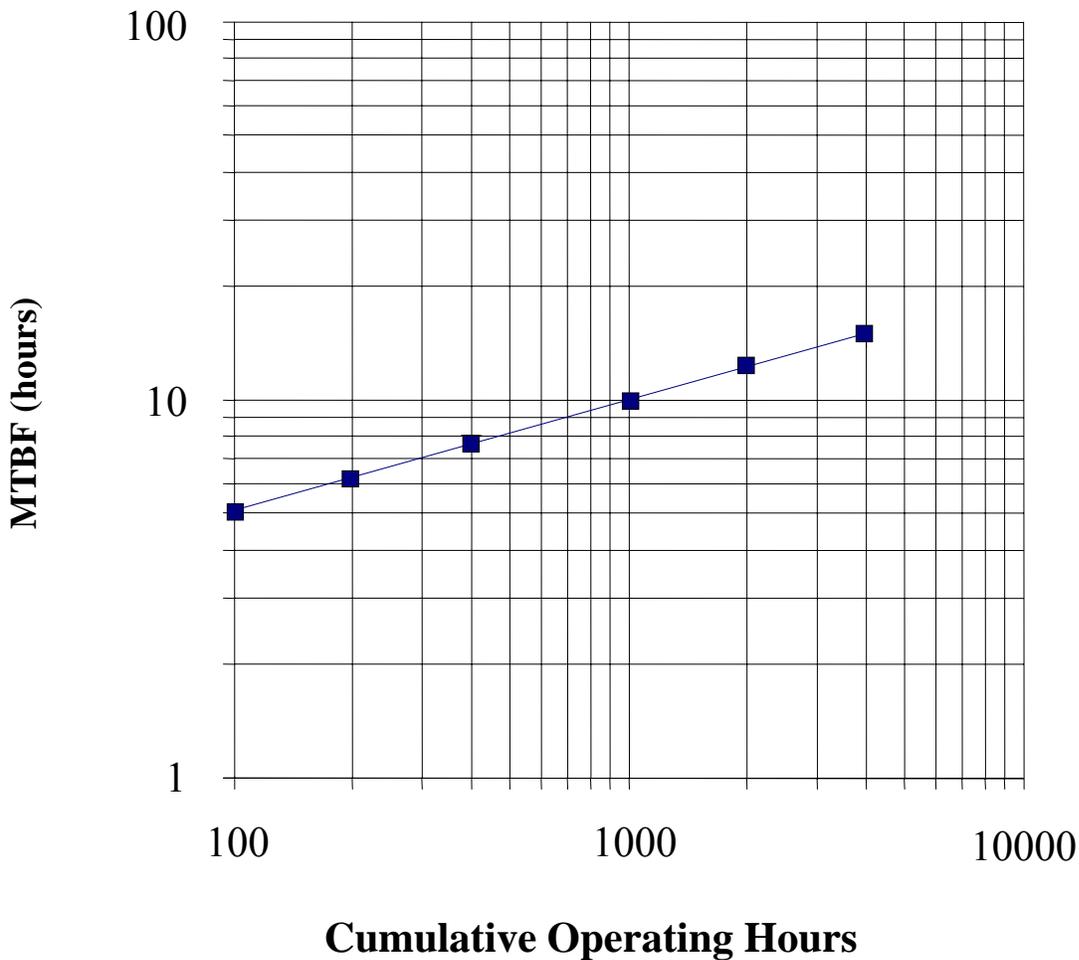
Figure 2



10.10. The Growth Rate - α

Duane's next (third) step was to draw a "best-fit" straight line through the data points and calculate the "slope" of the line. Duane then used the value of the slope of the line to develop relationships between test time and achieved reliability.

Figure 3 Line Fitted to Data



The slope, which we will designate with the symbol alpha (α), can be calculated as follows: Pick any two points on the straight line just drawn (e.g. 100 and 3800 operating or test hours, and 5 and 14.3 hours MTBF). Calculate α via the following formula:

$$\alpha = \frac{\log(\text{MTBF}_2 / \text{MTBF}_1)}{\log(\text{time}_2 / \text{time}_1)}$$
$$\alpha = \frac{\log(14.3/5)}{\log(3800/100)} = \frac{\log 2.86}{\log 38} = \frac{0.4564}{1.5798} = .29$$

10.11. Interpreting the Growth Rate

The slope has three valuable functions: (1) To provide a method to determine the current MTBF of an item, (2) to indicate the effectiveness of the TAAF process, and (3) to allow the development of a mathematical relationship between test time and MTBF.

➤ Current MTBF

The last MTBF value (14.3 hours at 3800 test hours) on the Duane log-log charts was calculated by dividing the total number of test hours accumulated to date by all the failures observed to date. This cumulative MTBF or $MTBF_C$ is an average MTBF over the period. Since the MTBF has improved with time, the current value should be greater than this average MTBF. If we go back to the original data, we see that for the last period, 1800 hours divided by 100 failures equals 18 hours MTBF. As expected, this value is greater than the average MTBF for the entire 3800 hours. Note, however, that this 18 hour value is itself an average. So the question remains: What is the MTBF after 3800 hours of test time?

The Duane model states that the current, or instantaneous MTBF ($MTBF_I$), is a simple function of $MTBF_C$ and α :

$$MTBF_I = \frac{MTBF_C}{1 - \alpha}$$

To determine the current MTBF at 3800 hours:

$$MTBF_I = \frac{MTBF_C}{1 - \alpha} = \frac{14.3 \text{ hours}}{1 - .29} = 20.4 \text{ hours MTBF}$$

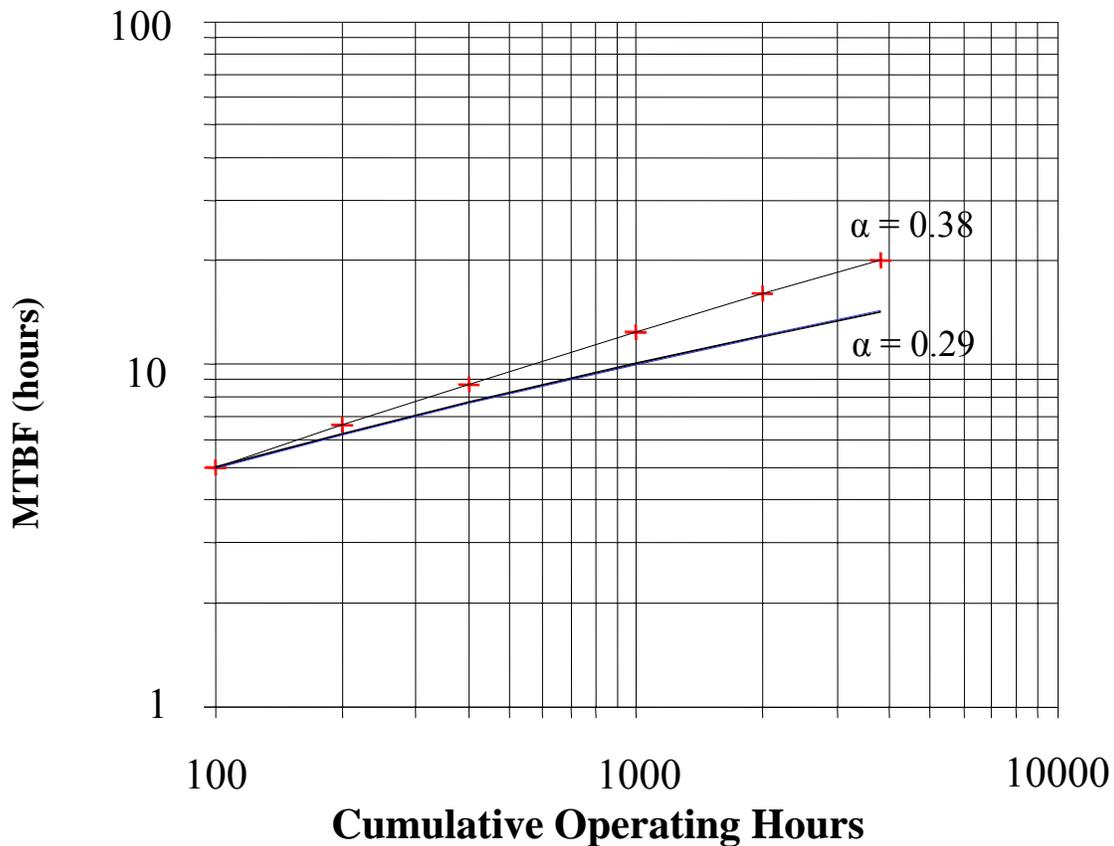
As expected, the current MTBF at 3800 hours is significantly greater than the 14.3 average value and is even greater than the 18 hour value observed over the last 1800 hours. The interpretation of this current MTBF is that (1) if no additional improvements are incorporated, (2) if there are no changes in the way the system is operated and maintained, and (3) if the system does not age, then the system will theoretically demonstrate a relatively constant 20.4 hours MTBF for the remainder of its useful life.

➤ Growth Data for a Second System

Period	Operating Hours - Period	Operating Hours - Cumulative	Failures in Each Period	MTBF in Each Period	Cumulative failures	Cumulative MTBF
1	100	100	20	5.0	20	5.0
2	100	200	10	10.0	30	6.7
3	200	400	16	12.5	46	8.7
4	600	1000	35	17.1	81	12.3
5	1000	2000	44	22.7	125	16.0
6	1800	3800	65	27.7	190	20.0

Figure 4 shows two cumulative data lines: One is of our original system; the second line represents the second system. Note that both systems were tested for 3800 hours and began with the same MTBF (5 hours). But note the steeper slope (0.38 vs. 0.29) and higher cumulative MTBF (20.0 vs. 14.3) at the end of the test.

Figure 4 Different Growth Rates



Using Duane's conversion formula, the second system achieved:

$$\frac{20.0 \text{ hours MTBF (cumulative)}}{1 - 0.38} = 32 + \text{ hours MTBF (instantaneous)}$$

vs. 20 hours for the initial system! Why the significant difference in achieved MTBF for the second system? The answer is that the TAAF process was more effective in the second system test program. In the first period, 20 failures were found in each test program. However, more effective failure analyses and corrective actions resulted in fewer failures occurring in the second and subsequent periods. The slope of the cumulative curve is therefore, an indicator of the effectiveness of the TAAF process.

Actual growth rates vary significantly between equipment, environments, and programs. But, typically, growth rates range from 0.1 (only critical failure modes are corrected, slowly) to 0.6 (most failure modes are corrected, quickly). One study (Ref 5) showed that (1) if **all systematic (repeat) failure mechanisms** are flagged for corrective action after the second occurrence, and (2) an effective fix is incorporated into the test items by a time equal to four times the MTBF of the uncorrected failure mechanism, the theoretical growth rate is 0.6. If **alternate systematic failure mechanisms** are effectively corrected under the same conditions, the growth rate is 0.23.

Higher growth rates allow us to:

1. Achieve higher levels of reliability for the same amount of test time, OR
2. Achieve a specific reliability goal in a shorter period of time.

If we look again at the data for the second system, we see that at 1000 cumulative test hours, the system has achieved a cumulative MTBF of 12.3 hours. Converting that into an instantaneous value results in an achieved reliability of 19.8 hours MTBF. Thus, we could achieve essentially the same instantaneous MTBF for the second system as achieved for the first but with only 1000 hours of testing vs. 3800. Those additional 2800 test hours represent additional costs and program schedule time - not desirable conditions. Question: How does a program manager plan for a TAAF program? How much time should he or she allocate? How much will it cost?

10.12 Relating Time and Alpha to Reliability Growth

The cumulative MTBF line, being a straight line on log-log paper, can also be expressed as an equation. Let's go back to our equation for calculating the slope of the cumulative line.

$$\alpha = \frac{\log (\text{MTBF}_2 / \text{MTBF}_1)}{\log (\text{time}_2 / \text{time}_1)}$$

If we are concerned about the total time (time₂) required to achieve a desired MTBF, let's rearrange this equation accordingly.

$$\log (\text{time}_2/\text{time}_1) = (1/\alpha) \log (\text{MTBF}_2/\text{MTBF}_1)$$

$$\log (\text{time}_2/\text{time}_1) = \log [(\text{MTBF}_2/\text{MTBF}_1)^{1/\alpha}]$$

Taking the anti-log of both sides:

$$\text{time}_2/\text{time}_1 = (\text{MTBF}_2/\text{MTBF}_1)^{1/\alpha}, \text{ so}$$

$$\text{time}_2 = \text{time}_1(\text{MTBF}_2/\text{MTBF}_1)^{1/\alpha} \text{ or}$$

$$\text{time}_2 = \text{time}_1 \left[\frac{\text{MTBF}_2}{\text{MTBF}_1} \right]^{1/\alpha}$$

Using the terminology in MIL-HDBK-189 (Ref 1),
time₂ (total operating time) is designated by "T,"
time₁ by "t₁"; and
MTBF₁ by "M_I." Thus,

$$T = t_1 \left[\frac{\text{MTBF}_2}{M_I} \right]^{1/\alpha}$$

The above equation quantifies the amount of operating time required to achieve MTBF₂. But MTBF₂ is a cumulative MTBF, and we are concerned about achieving an instantaneous or current MTBF (M_F). We must change the cumulative MTBF to a current MTBF using Duane's conversion formula.

$$\text{Current MTBF (M}_F) = \frac{\text{Cumulative MTBF}}{(1 - \alpha)}$$

Thus,

$$\text{Cumulative MTBF} = M_F (1 - \alpha).$$

Substituting this relationship for MTBF₂ our final formula is:

$$T = t_1 \left[\frac{M_F(1 - \alpha)}{M_I} \right]^{1/\alpha}$$

T = Total test time (including t_1) required to achieve M_F .
 t_1 = The initial amount of test time associated with an initial, cumulative MTBF (M_I).
Note: T includes the t_1 time. Therefore, the formal RD/GT time = $T - t_1$
 M_F = The required or desired level of instantaneous MTBF
 M_I = The initial cumulative MTBF that is observed at time t_1
 α = The growth rate

10.13. Duane Growth Model - Initial Characteristics

The above equation is simple to calculate, but where do we get the values for the variables t_1 , M_I , and α to plan a TAAF program for a new item? Fact: you will only know the true value of these variables after you have conducted the TAAF. However, it is possible to estimate values for each variable.

- t_1 and M_I . Duane observed the following relationship in his data sets: When t_1 was approximately equal to 50 percent of the mature, predicted MTBF value (or 100 hours, whichever was greater), M_I was approximately 10 percent of that same mature, predicted MTBF (e.g. via MIL-HDBK-217). With this relationship, Duane provided a "starting point" for planning a TAAF program.

NOTE 1: M_I could be interpreted two ways, either as a **cumulative** MTBF or as an **instantaneous** MTBF. In an actual program, the small amount of data collected during t_1 is generally insufficient to perform a reliability growth analysis and determine the **instantaneous** MTBF at time t_1 . Therefore, it is a logical assumption that M_I is an average or **cumulative** value. (MIL-HDBK-189 specifically defines M_I as a cumulative value.)

NOTE 2: The 10 percent estimate for M_I is an average, default value based upon experience from many programs. The actual value varies, with one study showing a range from 4 to 26 percent (Ref 6). The higher percentages might be expected for systems that use more mature technologies/components or for systems that have more intense reliability design/management efforts early in the program. If historical experience is available that indicates a more probable starting percentage - use it instead of the "default" values.

NOTE 3: If the initial MTBF achieved during an RD/GT is greater or less than the planned value of M_I , then the TAAF plan should be reevaluated.

- Alpha (α). As with t_1 and M_I , α is best estimated on the basis of past performance. There are many variables to be considered and the resulting value for α will be very subjective. Some of those variables are:
 - a. Actual reliability growth rates from previous programs. Historical data (ideally on similar equipment from the same manufacturer) is probably the best source for an initial α .
 - b. Relative maturity of the new item. Mature items generally have less potential for improvement.

c. Phase of program (development or fielded). A RADC study reported that many programs achieve a growth rate of between 0.3 and 0.5 for systems still in the development phase. For fielded systems, including warranted items, the growth rate drops to between 0.1 and 0.3 (Ref 7)

d. The similarity of the new item to existing items (e.g., technology, complexity, manufacturing techniques, etc). A Westinghouse report indicated the following relationship between cost/complexity and growth rate:

System Complexity	α
High cost, high technology	.5
Medium cost, state of the art	.4
Low cost, off the shelf	.3

e. The severity of the environment. Debugging is much more effective if testing is accomplished in actual or simulated operational vs. a benign environment. (Ref 6)

Environment	Relative Growth Rate
TAAF/Stimulating	.5
Simulated mission	.35
Benign	.2

More deficiencies are discovered in a shorter period of time, and some deficiencies may never be discovered in a benign environment. These hidden deficiencies will only be discovered when the equipment is fielded!

If testing in an operational or simulated operational environment is better than testing in a benign environment, is it possible that testing in an environment that is more severe than the operational environment be even more effective? Possibly. The answer is very dependent upon the design of the equipment and the severity of the environment. In general, a more severe environment will cause failure modes to appear more quickly. The problem is that the more severe environment may cause additional failure modes that may never occur in the field and may have a negative effect upon the conduct of the TAAF process. If more severe stresses are to be used, they should be increased in small increments and the reaction of the units on test should be observed carefully. The stresses can be increased until failure analysis indicates that the failure modes being observed are resulting from stresses beyond the ultimate design capability, and that these failure modes should not occur in the normal operational environment. This process is called step-stress testing.

f. Other factors include: Cost and schedule constraints, existence of a plan for a warranty, availability of test assets and facilities, the ability of the contractor to perform adequate failure analysis and develop/incorporate effective corrective actions, and perhaps most of all, the attitude of the program managers for the government and the contractor

10.14. Planning a Formal RD/GT Program - An Example

An item has a requirement to achieve 1500 hours MTBF before it can begin a reliability qualification test. The item has a predicted MTBF of 2500 hours. A TAAF program is being developed using an expected α of 0.35. How many test (operating) hours are required to achieve the desired MTBF?

$$M_F = 1500 \text{ hours}$$

$$\alpha = 0.35$$

$$t_1 = 50\% \text{ of } 2500 \text{ hours} = 1250 \text{ hours}$$

$$M_I = 10\% \text{ of } 2500 \text{ hours} = 250 \text{ hours}$$

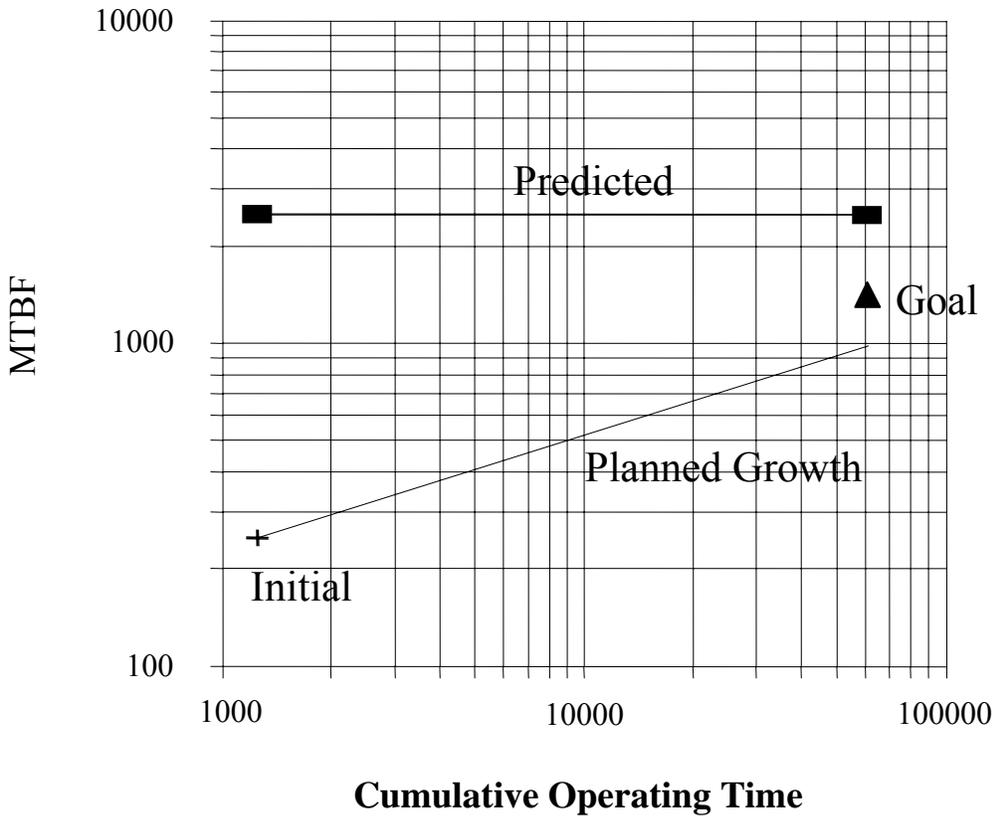
Using our formula:

$$T = t_1 \left[\frac{MF(1-\alpha)}{MI} \right]^{1/\alpha}$$
$$T = 1250 \left[\frac{1500(1-0.35)}{250} \right]^{1/0.35}$$
$$T = 1250 \left[\frac{975}{250} \right]^{2.857} = 1250(3.9)^{2.857}$$
$$= 61,047 \text{ operating hours (OH)}$$

Note: T includes the t_1 time. Therefore, the formal RD/GT time = $T - t_1 = 61,047 - 1,250 = 59,797$ oh

The above reliability growth plan is shown graphically in Figure 5.

Figure 5: A Reliability Growth Plan – based on the Duane Model



Note that the slanting line represents the expected growth in the **cumulative** MTBF, not the **instantaneous**. Also note that "T" includes all operating hours, including the initial test hours "t₁".

The number of operating hours for any proposed growth plan can be calculated via the above formula, or a rough estimate of the test time can be calculated via the table in the Appendix at the end of this chapter.

10.15. Calendar Time to Achieve a Desired Level of Reliability

If we were to place a single unit on test, it would only take 61,047 hours ÷ 8760 hours/yr. or 6.97 years to achieve the 1500 hours MTBF! Actually it would take longer. The above equation is based on operating hours. During an actual TAAF, the equipment will be turned off for: (1) evaluation/repair of the test item, (2) repair/adjustment of the test chambers/facilities, and (3) periods when the equipment is being cooled during a temperature cycle. For one radar program, the planned test efficiency was only 33% (Ref 5). Thus, for every hour the equipment is actually operating, it can be off for an additional two hours. Therefore, in our example, the

equipment would have to be in the TAAF program for 61,047 x 3 or 183,141 calendar hours or 20.9 years.

Obviously no manager is willing to wait 13.7 years to make a decision on a program! The test time must be shortened. We will look at three general approaches to reducing the calendar time:

- Increasing the number of test assets
- Improving the effectiveness of the test analyze and fix process
- Improving the inherent reliability of the item

➤ Increasing the Number of Test Assets

Increasing the number of items in the test is beneficial in that the calendar time is inversely proportional to the number of items on test. The more items - the shorter the time. Another benefit of putting additional items on test is that additional failure modes can be observed and corrected.

If, in our example, we were to put ten items on test, the total test time would be reduced to $13.7 \alpha 10$ or 1.4 years. An obvious improvement, but over a year is still a long time. And, we must set aside funds for ten units vs one. We may even require additional test equipment in order to monitor multiple test items. So, just putting additional units on test may not be the best answer.

➤ Increasing the Effectiveness of the TAAF Process - α

As discussed earlier, α is an indicator of the effectiveness of the TAAF process. If the emphasis on finding and fixing problems was increased, this would appear as an increase in α .

If we assume α can be improved to 0.5,

$$T = 1250 \left[\frac{1500(1 - 0.5)}{250} \right]^{1/0.5}$$

$$T = 1250 \left[\frac{750}{250} \right]^{2.0} = 1250 (3)^2$$

$$= 11,250 \text{ total operating hours, or } 11,250 - 1250 = 10,000 \text{ RD/GT operating hours}$$

$$= 10,000 \times 2 = 20,000 \text{ calendar hours} = 2.3 \text{ years (with only one test unit)}$$

Increasing the growth rate obviously improved the program schedule, but 3.9 years is still too long. And, as we will discuss in section 10.16, increasing the growth rate requires additional effort and expense.

➤ Improving the Inherent Reliability of the Item

The third approach to reducing the calendar time is to improve the predicted reliability by improving the inherent design of the item. To illustrate, we will assume that the design can be improved and the predicted MTBF increases to 4000 hours. This means that

$$t_1 = 50\% \text{ of } 4000 \text{ (vs } 2500) \text{ hours} = 2000 \text{ hours}$$

$$M_1 = 10\% \text{ of } 4000 \text{ (vs } 2500) \text{ hours} = 400 \text{ hours}$$

$$T = 2000 \left[\frac{1500(1 - 0.35)}{400} \right]^{1/0.35}$$

$$T = 2000 \left[\frac{975}{400} \right]^{2.857} = 2000(2.4375)^{2.857}$$

$$= 25,503 \text{ (vs } 61,047) \text{ total operating hours, or}$$

$$= 25,503 - 2000 = 23,503 \text{ RD/GT oh}$$

$$= 23,503 \times 2 = 47,006 \text{ calendar hours or } 5.4 \text{ years if we use only a single test item.}$$

➤ Combining Techniques

Let's combine the RD/GT reduction techniques as follow:

Number of test units = 5 (less costly than 10)

Projected growth rate = 0.4 (more conservative than 0.5)

Predicted MTBF = 3000 hours. (more achievable than 4000 hrs)

Then: $t_1 = 50\% \text{ of } 3000 \text{ hours} = 1500 \text{ hours}$

$M_1 = 10\% \text{ of } 3000 \text{ hours} = 300 \text{ hours}$

$$T = 1500 \left[\frac{1500(1 - 0.4)}{300} \right]^{1/0.4}$$

$$T = 1500 \left[\frac{900}{300} \right]^{2.5} = 1500(3)^{2.5}$$

$T = 23,383 \text{ operating hours, or } 21,883 \text{ RD/GT hours}$

$$= 21883 \times 2 \text{ or } 43,766 \text{ calendar hours}$$

With 5 units on test,

$$\text{Total calendar time} = 43,766 \div 5$$

$$= 8753 \text{ calendar hours or approximately } 12 \text{ months.}$$

A year is still a long time. Additional iterations may be necessary to arrive at a more reasonable time to conduct a formal RD/GT

10.16. Cost of Achieving a Reliability Objective in Less Time

Improving the reliability is not without some additional initial cost. If additional units are to be tested, they must be built. More environmental chambers, test/monitoring equipment and/or facilities may be required. If the TAAF process is to be more effective, additional engineering effort is required to carefully analyze more failure modes, to develop additional corrective actions, and/or do everything quicker. If the design is to be improved, this also requires additional engineering effort and/or higher quality/more expensive parts/materials. The bottom line is that improving the achieved reliability of an item in development will probably result in an increased cost for the production units. This may, however, be offset by improved readiness/performance and decreased logistics resource requirements in the field!

10.17. Achieved MTBF for a Given Number of Test Hours

Another way to look at these reliability growth relationships is to ask the question: "What reliability can I achieve if the item is tested for x hours or y more hours?" To answer this question, we can rearrange the equation we have been using into the following form:

$$M_F = \frac{M_I}{(1 - \alpha)} \left[\frac{T}{t_1} \right]^\alpha$$

If we take our last example system and only test for 3000 operating hours, what MTBF can we expect to achieve?

$$\begin{aligned} M_F &= \frac{200 \text{ hours}}{(1 - .4)} \left[\frac{3000 \text{ hours}}{1000 \text{ hours}} \right]^{0.4} \\ &= \frac{200}{.6} (3)^{.4} = 517 \text{ hours, less than our goal of 1000 hours MTBF.} \end{aligned}$$

Again, if we improve the inherent design potential (predicted MTBF) and/or the effectiveness of the TAAF process (indicated by α), then the achieved MTBF will be greater. For example, if we increase α to 0.45, the achieved MTBF =

$$\begin{aligned} M_F &= \frac{200}{(1 - .45)} \left[\frac{3000 \text{ hours}}{1000 \text{ hours}} \right]^{0.45} \\ &= \frac{200}{.55} (3)^{.45} = 596 \text{ hours, more than the first value of 517 hours MTBF.} \end{aligned}$$

10.18. Projecting Growth into the Future

Consider the situation where an item has already demonstrated a certain reliability, either through a formal TAAF process or maybe as the result of field use. Can it be improved further? Possibly. For example, assume we have a system that accumulated 50,000 flight hours in the field, had a growth rate of only 0.1, and has a current, instantaneous mean time between maintenance (MTBM) of 50 hours. Management decides to increase the emphasis on finding and fixing problems. Assuming a new growth rate of 0.25 for the next 20,000 hours, what new level of reliability can be expected at the end of this additional time?

$$M_F = \frac{M_I}{(1 - \alpha)} \left[\frac{T}{t_1} \right]^\alpha$$

Looking at our formula above, we have all the information we need, except M_I . M_I is a cumulative value and all we have is an instantaneous value of 50 hours.

$$50 \text{ hours} = \frac{M_I}{1 - \alpha} \text{ or,}$$
$$M_I = 50(1 - \alpha) = 50(1 - .1) = 45 \text{ hours}$$

Substituting into our original equation, we get:

$$M_F = \frac{45}{(1 - 0.25)} \left[\frac{70,000}{50,000} \right]^{0.25} = 60(1.4)^{0.25}$$
$$= 65.3 \text{ hours MTBM}$$

Note that "T" is equal to the sum of the additional hours planned AND all the previously accumulated test time. When performing a reliability growth assessment on an already existing system, it is important to always include ALL the previous operating hours.

10.19. Managing a Formal RD/GT Program

Managing a formal RD/GT program requires much advanced planning. Consideration should be given to applying TAAF during the EMD (engineering and manufacturing development), production, and fielded phases. The RD/GT should be integrated into the overall program schedule. Funds must be allocated for additional test assets, test equipment, engineering analysis/design effort, etc. It also requires development of detailed operating instructions concerning the conduct of the TAAF to include: detailed reporting of malfunctions, thorough and timely failure analysis, and careful verification of effective corrective actions. Tasks associated with a reliability growth program should be specified in the contractual statement of work. Following is a brief example of a SOW task for reliability growth:

"Four preproduction units shall be placed in environmental chambers and functionally operated for a period of 1000 operating hours. During this period, the four units in the chambers will be subjected to thermal cycling, random vibration, and humidity in an attempt to discover design weakness. As malfunctions occur, associated data shall be entered into a closed-loop failure reporting, analysis and corrective action system (FRACAS). The malfunctioning units will be troubleshot in order to isolate the cause, and the malfunctioning unit(s) will be removed and sent to a shop for additional diagnosis and evaluation. Other test units will continue to operate. As soon as possible, each malfunctioning unit will be repaired and returned to the chamber for continued TAAF testing. As fixes are incorporated in the test items, additional hours will be accumulated to verify the effectiveness of the fixes and identify any new failure modes. As additional malfunctions occur, trends and patterns will be analyzed by the failure review board. The effectiveness of the TAAF process will be tracked against the approved growth plan."

It is not the intent of this chapter to discuss the many details of managing an RD/GT program. Good sources for such information are:

- MIL-HDBK-189, Reliability Growth Management (Ref 1)
- MIL-STD-785, Reliability Program Plan (Ref 8).
- The TAAF Process - A Technical Brief For TAAF Implementation, Jan 1989 (Ref 9).

10.20. Other Reliability Growth Models

The primary tool discussed in this chapter is the "Duane" model. It is important to know that the Duane model is only one of many mathematical relationships that have been developed to characterize the reliability growth process. Other models include: AAMSA (developed by the US Army, Ref 1), IBM, and Lloyd-Lippow. Each of these models has advantages and disadvantages. One model may mathematically model one type of system better than another model. Some models have the ability to address both "latent defects" and "random" failures. However, there is a significant reason we have not addressed these other models in this chapter.

Most other models are dependent upon the availability of detailed experience data, especially the time-to-fail for each observation. For most situations in the Air Force, we do not have this detailed data. We generally only have time and failure data aggregated over intervals of time (e.g., monthly). The Duane model may not always be the best model to use, but it is one that can be used with aggregate data.

10.21. Summary

In this chapter we have discussed the TAAF process as a means to achieve a reliability objective. However, TAAF is not a stand-alone process. Good design must be coupled with a TAAF in order to have a reasonable chance of meeting a stated objective. In other chapters in this textbook, you will also discover how the TAAF process can be applied to other related activities, namely Reliability Qualification Testing, Production Reliability Acceptance Testing, and Environmental Stress Screening

References

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6. RADC-TR-86-148, "Reliability Growth Prediction"
7. RADC-TR-84-20, "Reliability Growth Testing Effectiveness"
8. MIL-STD-785, "Reliability Program for Systems and Equipment; Development and Production"
9. "The TAAF Process - A Technical Brief for TAAF Implementation," Jan 89.

Appendix A

Operating time required to achieve MTBF objective (Duane model)
(in multiples of predicted MTBF)

Objective as Percent of Predicted MTBF	Planned (expected) Growth Rate (α)					
	0.1	0.2	0.3	0.4	0.5	0.6
100	1.74x10 ⁹	16.4K	328	44.1	12.5	5.04
80	1.87x10 ⁸	5369	156	25.2	8.00	3.47
60	1.05x10 ⁷	1274	59.8	12.3	4.50	2.15
50	70x10 ⁶	512	32.5	7.79	3.13	1.59
40	183K	168	15.5	4.46	2.00	1.09
25	1663	16.0	3.23	1.38	.781	.500

1. For a predicted MTBF (MTBR, MTBM, etc.) **less than or equal to 200 hours**, the total number of operating hours is equal to the appropriate factor from the table above **times 200**.

Example: Predicted MTBF = 80 hours
 M_F goal = 48 hours (60% of predicted)
 Expected growth rate = 0.3
 Factor from table = 59.8

Total test time = 59.8 X 200 = 11,960 operating hours.
 initial, cumulative MTBF (M_I). **Note: T includes the t_1 time. Therefore, the formal RD/GT time = $T - t_1 =$**

$$11,960 - (50\% \times 200[\text{not the predicted in this case}]) \\ = 11,860 \text{ operating hours}$$

2. For predicted MTBF **greater than 200 hours**, multiply the above table value by the **predicted MTBF**.

Example: Predicted MTBF = 3000 hours
 M_F goal = 1500 hours (50% of predicted)
 Expected growth rate is 0.4

Total test time = 7.79 X 3000 = 23,370 operating hours.
 Formal RD/GT time = 23,370 - (50% x 3000[predicted])
 = 21,870 operating hours

Note: The times calculated via this table are the total planned **operating or "on" hours** for the equipment undergoing RDGT/TAAF. The total planned **calendar time** will be a function of the ratio of total hours to "on" hours and the number of units on test.

Chapter 11

RELIABILITY TESTING

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SYMBOLS, TERMS AND EQUATIONS USED IN RELIABILITY TESTING

OC Curve = Operating Characteristic Curve

RDGT = Reliability Development Growth Testing

TAAF = Test Analyze and Fix

Test Conditions

α = the chance of rejecting a system whose MTBF is Θ_0 .

$1-\alpha$ = the chance of accepting a system whose MTBF is Θ_0 .

β = the chance of accepting a system whose MTBF is Θ_1 .

β = 1.0 - confidence stated in the specification

$1 - \beta$ = Confidence that the system MTBF (or reliability) is at least Θ_1 (or lower test Reliability) if the reliability test is passed

$1 - \alpha$ = Confidence that the system MTBF (or reliability) is at least Θ_0 (or upper test Reliability) if the reliability test is passed

Θ = possible MTBFs for the system

P_a = probability of acceptance.

R_1 = lower test reliability

R_0 = upper test reliability

P_1 = lower test unreliability

P_0 = upper test unreliability

11.1. Introduction

Testing should be considered as a means of uncovering weaknesses in either the design process or the production process. During the design process it begins as soon as small assemblies are available and continues through each stage of assembly culminating with reliability qualification test (RQT).

During the production stage testing begins with the parts that are either manufactured in house or purchased from vendors and continues through the assembly of the product concluding with the production reliability acceptance test (PRAT). It is important to ensure that the assemblies be free of defects before proceeding to the next stage of production. It is also important that processes be operator controllable, where operators:

- have all the knowledge they need to do their work,
- are provided with equipment that is capable of meeting the specs,
- have adequate resources,
- get feedback on their efforts,
- work in an environment conducive to producing quality products,
- understand that quality is important and see quality visually demonstrated by the leaders in the organization.

This section begins with an overview of the various tests that occur during development and production but most of the section deals with the Reliability Qualification Test (RQT), the Production Reliability Acceptance Test, (PRAT), and the Quality Assurance testing during production. RQT and PRAT differ only in their application, RQT is conducted during development and PRAT is conducted during production.

A summary of the different kinds of tests that could be conducted include:

Pre-Production:

- Prototype Tests
- Breadboard Tests
- Accelerated Tests
- Reliability Development Growth Tests (RDGT) aka (TAAF)
- Environmental Tests
- Reliability Qualification Tests (RQT)

Production:

- Receiving Inspection Tests
- First Article Tests
- Quality Assurance Tests
- Performance Tests
- Production Reliability Acceptance Tests (PRAT)
- Packaging Tests

Life Tests

11.2. Pre-Production Tests

Pre-production tests are conducted for the purpose of enhancing the equipment design or to demonstrate a requirement.

1. Prototype and breadboard tests are conducted early in the development stages to determine the status of the design and design changes.
2. RDGT is used to help grow the MTBF to an acceptable level, hopefully to the design level. RDGT is discussed in another chapter.
3. Environmental tests are conducted to insure that the equipment will not fail in various environments such as, sand and dust, immersion, shock, explosive decompression, etc. These tests are usually conducted using very small sample sizes. For some products these tests are also conducted on production units, e.g., automobiles are driven through a water spray to check for leaks.
4. Accelerated tests are conducted under greater stress to shorten the length of a test. These tests are somewhat unusual and are used primarily when the time to test under normal conditions would be extremely lengthy. Accelerated tests are often used to determine the life of a system. For these tests it is critical that the relationship between the accelerated test results and normal test results be known. Accelerated tests could also be conducted with production units.
5. The Reliability Qualification test is conducted to demonstrate that the requirement (MTBF or failure rate) in the equipment specification has been achieved in the development process.

11.3. Production Tests

The tests on the second half of the list are conducted on production units to ensure that a requirement is met.

1. Receiving Inspection tests are performed to check dimensions and other engineering requirements on parts that have been purchased from a vendor or manufactured by the organization. The purpose of a Receiving Inspection (RI) test is to ensure that products purchased from vendors meet their specifications. However, in some organizations products produced within a plant also pass through RI; in that case the production process is viewed as a supplier to the organization. The decisions needed before initiating the tests are similar to those listed for QA.

For RI, the lot size is the amount received from the vendor.

2. First Article tests can have different meanings. In the sense used in this text, first article refers to the tests or inspections made on the first three to five units that come off the production line. The tests can be conducted on parts, assemblies or completed units. The purpose of these tests is to qualify the equipment and the procedure used to produce the units being tested.

3. Quality Assurance tests are primarily inspections made during the production process on parts, assemblies, or final products. These tests are conducted to insure that tolerances are met, and can be conducted by production operators or quality assurance inspectors.

4. Performance tests are conducted to make sure each unit actually works, and meets all other requirements. This is a 100% test and is often done by production personnel.

5. Production Reliability Acceptance Tests (PRAT) are similar to the RQT tests done at the end of the development phase, except that production units are used. The purpose of these tests is to ensure that the production units meet the requirements.

6. Packaging Tests are conducted to determine how well the unit being shipped will survive handling and shipment to its final destination.

7. Life tests are performed to determine when the equipment reaches wearout if the equipment is electronic; for mechanical equipment the life test is performed to determine when the failure rate reaches a specified value. The life of an equipment is generally not predictable unless historical data for similar units is available.

11.4 Other Kinds of Testing

Tests can also be classified in other ways such as:

- Destructive vs. Non-destructive tests
- Ambient vs Environmental
- Actual conditions vs. Simulated

DESTRUCTIVE vs. NON-DESTRUCTIVE: The tests listed above can be destructive or non-destructive depending on the characteristic being tested. Testing a fuse, and testing bullets are destructive tests because the units being tested are destroyed in the test. Other destructive tests include testing the tensile strength of metal, pressure testing cans, pulling welded joints apart, tasting food, etc. Destructive tests are used when there is no other way to test the characteristic of a product. There

can be more sampling risk with destructive tests because sample sizes are usually small.

In a non-destructive test the unit is not destroyed. Instead, measurements that are taken show if the product's characteristics are being met. Non-destructive testing is usually less expensive and larger sample sizes can be taken which reduces the sampling risks.

AMBIENT vs. ENVIRONMENTAL: Ambient vs. environmental refers to the conditions under which the tests are conducted. For RDGT, RQT, and PRAT it is recommended that the tests be conducted under specified environments that duplicate as much as possible the environments anticipated in use.

Ambient testing usually includes testing under static conditions when doing bench testing, checking part dimensions, etc. These tests are usually cheaper and quicker than environmental tests.

DISCRETE OR VARIABLE TESTS: Some products can only be tested on a pass/fail criteria; these tests are called discrete tests and are based on the Binomial or Poisson distribution. They require large sample sizes but it is possible to combine several characteristics in one test. Control charts (np , p , c , and u) can be used to determine the stability of discrete processes.

Variable tests deal with measurements taken with some type of gauge, e.g., a watch, caliper, volt meter, thermometer, etc. These tests provide more knowledge about the product than discrete tests but take more time. However, they do not require sample sizes as large as the discrete tests. The data for these tests are often based on the Normal distribution, hence \bar{X} -bar and R control charts can be used to determine process stability. Variables tests are usually preferred over discrete tests because they provide more knowledge about the product and the process producing the product.

Both discrete and variables tests can occur at any level in the production of the system.

ACTUAL CONDITIONS vs. SIMULATED: Environmental tests can be simulated or tested under actual conditions. Actual conditions are usually preferred but not always feasible. The decision as to which to use depends on such factors as:

- size of the unit,
- nature of the unit,
- frequency of testing,
- complexity of the instrumentation,
- complexity of the test,
- accessibility of the natural environment,
- relative costs, and

- relative time.

11.5 Levels of Tests

Tests performed as part of the production process can occur at various levels. The levels are: part (P), subassembly (SA), assembly (A), subsystem (S) and system (S). Some form of testing is performed at each level based on the capability of the processes at each level. Processes that are less capable require more frequent testing. Hence, efforts should be made to reduce process variation so that testing can be reduced if not eliminated.

The types of tests and the test levels that typically occur during the production process are as follows:

TYPE OF TEST:	LEVEL:
Receiving Inspection Tests	Part, but could be higher
First Article Tests	All
Quality Assurance Tests	All
Performance Tests	System
PRAT	System
Operational Tests	System
Packaging Tests	System
Life Tests	System

11.6 Procured Units

Units that are purchased can be tested at the seller's plant or they can be tested as part of the buyer's Receiving Inspection operation. Ideally a relationship needs to be developed between the buyer and the seller so that mutual trust, and cooperation grow and prosper; and that both parties work continuously to reduce process variation. If this happens both parties will benefit. In addition to trying to reduce process variation, the seller may also be controlling the process at the source to minimize future inspection.

11.7 Types of Tests and the Production Process

The types of tests used as the product flows from the part stage to the final assembly or system stage is as follows:

Production Process	Type of Test
Parts:	
- manufactured	Quality Assurance*
- purchased	Receiving Inspection

Subassembly	Quality Assurance
Assembly	Quality Assurance
Subsystem	Quality Assurance
System	PRAT

* For Quality Assurance tests, various ways can be used to inspect the process and/or the product including: dimensional checks, go/no-go tests, lab tests, and visual inspections. Control charts can be used at any level of testing.

11.8 Quality Assurance Tests

Quality Assurance (QA) tests are conducted to ensure that specifications are met during the production process. The tests can be conducted by production operators or QA personnel.

The tests can be discrete or variable and require the following types of decisions in preparation for the tests:

- purpose of the test,
- characteristics to be tested,
- definition of a defective and a defect,
- individual to perform the test,

- definition of a lot,
- sample size,
- acceptability criteria,
- sampling risks,
- frequency of testing,
- inspection points,
- inspection level,

- procedure to follow if the test is not passed,
- procedure for disposing of rejected items,

- inspection equipment required to perform tests,
- calibration procedure for inspection equipment,

- use of control charts,
- process capability determination,

- type of records to keep,
- function that will maintain the records,

- level where ESS is applied

The purpose of this checklist is to help organize the role of testing during the stages of the production process. Ideally

these decisions are documented and made part of a written test procedure.

11.9. Testing Program

The tests conducted during the production process are actually only a part of the overall testing program which also includes development testing, a corrective action plan, environmental stress screening (ESS), a calibration program, data collection and record keeping, and related training.

11.10. Test Management

In managing the testing program it is better if one organization is responsible for the entire program rather than have the responsibilities divided between several functions. However, it will be crucial that there be no communication barriers between the parts of the organization involved in testing, including the activities during development and vendor operations.

11.11. Test Planning

It is also important that the entire testing program be planned and designed before any testing is performed. Test plans have to be developed, operators trained, equipment calibrated, instructions written, manuals and forms developed, and everyone who is involved needs to be educated in what is expected of them.

11.12. Testing Procedure for RQT and PRAT - Exponential Distr.

A Reliability Qualification Test (RQT) and a Production Reliability Acceptance Test (PRAT) are similar tests. The RQT is conducted at the end of the Development Phase to evaluate the status of the equipment. If the development process has been successful the equipment should have a high probability of passing the RQT, which would imply that the equipment meets the specification.

A PRAT is conducted on a periodic basis to demonstrate that each production lot meets the same specification. Hence, it is possible that the same test plan could be used for RQT and PRAT; but there could be a logical reason not to use the same test plan. In either case the mechanics and decisions to be made are essentially the same except for the definition of a failure.

To conduct RQT and PRAT tests decisions must be made in the following areas: (RADC)

1. Identify the purpose and scope of the test:
 - overall test objectives
 - description of test to be performed
 - list of reference documents
2. Describe the test facilities.
3. Describe the test requirements:
 - ESS (cycles, temperature change rate)
 - test length (test time or number of failures)
 - number of test units (with or without replacement)
 - accept and reject criteria
 - test plan (test time, number of allowable failures)
 - government furnished equipment (GFE)
4. Develop a schedule for the test.
5. Identify the test conditions (profile):
 - thermal cycle
 - vibration survey
 - on/off schedule
 - preventive maintenance considerations
 - duty cycle
6. Identify who will monitor the test.
7. List the organizations/individuals who will participate.
8. Develop a definition of failure:
 - design defects
 - manufacturing defects
 - physical or functional degradation below specs
 - intermittent or transient failures
 - failures of limited life parts which occur before specified life of the part
 - failure which cannot be attributed to a specific cause
 - failure of built in test (BIT)
 - non-relevant failures:
 - improper installation or handling
 - external instrumentation or monitoring equipment
 - overstressed beyond spec limits due to a test facility fault
 - procedural errors introduced by the technicians
 - failures induced by repair actions
 - secondary failures which are a direct result of a failure of another part within the system
9. List the ground rules for the test.
 - classification of failures
 - pattern failures
 - malfunctions observed during troubleshooting/repair
 - test time accumulator

- failure causes (system or part: design and/or quality)
- action taken following failed tests

10. Describe test logs:

- equipment data sheets
- narrative record of required test events
- failure summary record
- failure report

11.13. Developing an Exponential Test Plan (Item #3 above)

An exponential test plan consists of a test time and an acceptance criteria. The test plan is determined by four factors, the upper test MTBF, the lower test MTBF, the producer's risk, and the consumer's risk.

PRODUCER'S RISK AND Θ_0 : The **producers risk**, (alpha, α), is the chance of rejecting systems with MTBFs at the upper test level of the MTBF (theta-zero, Θ_0); the chance of accepting systems at the upper test MTBF is $1 - \alpha$. Alpha is usually .10 or 10%.

For example, if Θ_0 is 1000 hours and the system MTBF is also 1000 hours, the chance the equipment will pass the test is $1 - \alpha$. If α is .10 then the chance of passing the test is .90 or 90%.

CONSUMERS RISK AND Θ_1 : The **consumers risk**, (beta, β) is the risk of accepting systems with MTBFs at the lower test level of the MTBF, (theta-one, Θ_1). Beta is also usually .10. Beta can be determined from the specification if the specification is written with a confidence statement. [Beta = 1.0 - Confidence in the spec].

For example, the specification, "must demonstrate 94% reliability with 90 % confidence," implies that beta is .10 or 10%, because $\beta = (1.0 - \text{Confidence}) = (1.0 - .9)$.

Let us now suppose that the system MTBF is 300 hours, that $\Theta_1 = 300$ hours and $\beta = .10$, then there is a 10% chance that the equipment will pass the test. Therefore to have high chance of passing the test, the MTBF for the system being tested must be increased.

Summary of symbols:

α = the chance of rejecting a system whose MTBF is Θ_0
 $1 - \alpha$ = the chance of accepting a system whose MTBF is Θ_0

β = the chance of accepting a system whose MTBF is Θ_1 and is found by subtracting the confidence in the spec from

1.0

Θ = possible MTBFs that the system could have

P_a = probability of acceptance.

The four factors, Θ_o , Θ_1 , α , and β are usually specified in some manner. The lower test MTBF and β can be found in the equipment specification; the upper test MTBF and α , if not in the spec, may be identified by the test plan to be used for the test.

On the other hand, what if you are the one who has to specify α and the upper test MTBF, how do you do it? The choice of the upper MTBF depends on four things: (See Example 2.)

- the scheduled time available for testing,
- the number of units available for testing,
- the chance of accepting the MTBF that has been achieved in the growth process, and
- the producer's risk.

SPECIFIED MTBF: In reliability testing, Mil-HDBK-781 defines the specified MTBF as the lower test MTBF, or minimum acceptable MTBF. This means that a system that is designed to meet the spec will have a low chance of passing the test, (or a low chance of acceptance). Therefore, for a system to pass a Mil Std 781 test, the system must be designed to an MTBF higher than the specified MTBF.

DISCRIMINATION RATIO (DR): The discrimination ratio is the ratio of Θ_o to Θ_1 and is found by dividing Θ_o by Θ_1 , that is,

$$DR = \Theta_o / \Theta_1. \quad (11.1)$$

A discrimination ratio close to 1.0 (say between 1.05 and 1.2) implies that Θ_o and Θ_1 are relatively close to each other; in this situation the total test time will be a large number.

On the other hand, if the ratio of Θ_o to Θ_1 is about 3.0, the total test time will be relatively small. Hence, in designing a test plan it is important to pay close attention to the value of the discrimination ratio.

It is not unusual to have DR in the range of 2.0 to 3.0, but the choice of a DR depends on other factors, such as time available for testing, cost of testing, the number of units available for testing, and the probability of passing the test that is desired.

DEVELOPING A TEST PLAN WHEN Θ_o , Θ_1 , α , and β ARE KNOWN:

To develop a test plan the following steps are recommended:

1. Determine values for α , β , θ_0 , and θ_1 .
2. Compute the DR, where $DR = \theta_0/\theta_1$.
3. Using the table below find the appropriate column for α and β .
4. Proceed down that column until you come to the DR computed in step 2. Note: If the exact DR is not available, you may pick the closest DR in the table, or the first DR in the table that is less than the calculated DR. By taking a smaller DR from the table you are increasing total test time, thereby reducing the sampling risk.
5. In this row read the "Total test time (Multiples of θ_1)" and the acceptance number, C. Multiply the test time multiplier by θ_1 to get the total test time.

Fixed Length Test Plans, extracted from Figure 18 (10 Percent Consumer's Risk (β) Test Plans) on page 236 of MIL-HDBK-781A, are as shown in the table below:

$\beta = .10$					
C (# of Failures Allowed)	Total Test time (Multiples of θ_1)	DRs for $\alpha =$			
		.05	.10	.20	.30
0	2.30	44.88	21.85	10.32	6.46
1	3.89	10.95	7.31	4.72	3.54
2	5.32	6.51	4.83	3.47	2.78
3	6.68	4.89	3.83	2.91	2.42
4	7.99	4.06	3.29	2.59	2.20
5	9.27	3.55	2.94	2.38	2.05
6	10.53	3.21	2.70	2.22	1.95
7	11.77	2.96	2.53	2.11	1.86
8	12.99	2.77	2.39	2.02	1.80
9	14.21	2.62	2.28	1.95	1.75
10	15.41	2.50	2.19	1.89	1.70
11	16.60	2.40	2.12	1.84	1.66
12	17.78	2.31	2.06	1.79	1.63
13	18.96	2.24	2.00	1.76	1.60
14	20.13	2.18	1.95	1.72	1.58
15	21.29	2.12	1.91	1.69	1.56
16	22.45	2.07	1.87	1.67	1.54
17	23.61	2.03	1.84	1.64	1.52
18	24.76	1.99	1.81	1.62	1.50
19	25.90	1.95	1.78	1.60	1.49

6. The test plan is: Put "n" units on test; run the test until the total test time (from step 5) has been accumulated; if the total number of failures is equal to or less than the acceptance number (from step 4), the units being tested have passed the test.

Example 1: A specification calls for an MTBF of 100 hours* to be demonstrated with 90% confidence with an upper test MTBF of 300 hours and a producer's risk of 5%. What test plan could be used?

1. $\theta_0 = 300$ hours and $\alpha = .05$
 $\theta_1 = 100$ hours and $\beta = .10$
2. DR = $300/100 = 3.0$
3. Find the column for $\alpha = .05$ and $\beta = .10$
4. The closest DR is 2.96
5. The test time multiplier is 11.77 and the acceptance number is 7. Total test time is:
 11.77×100 hours = 1177 hours

6. Conduct the test until 1177 hours have been accumulated; if 7 or fewer failures occur the equipment is accepted.

*Note: If the specification is given as a reliability, it is necessary to convert the reliability to an MTBF using the following equation:

$$\text{MTBF} = -(\text{mission time})/\ln(\text{Reliability}) \quad (11.2)$$

Developing A Test Plan When Only θ_1 and β Are Known And Testing Time Is Critical:

To find a test plan when θ_1 , and β are known and the test time is critical it is necessary to pick a value for α and then either assume a value for θ_0 and proceed as described above, or pick a DR and multiply it by θ_1 to find θ_0 , since:

$$\theta_0 = \theta_1 \times \text{DR} \quad (11.3)$$

This method has application when the "Total Test time (Multiples of θ_1)", i.e., the multiplier, is being used to select a test plan. When test time is critical it is sometimes easier to find a test plan by first selecting the test time that is feasible and converting that test time to a "Total Test time (Multiples of θ_1)" multiplier. Then the upper test MTBF, θ_0 , is found by multiplying the DR (from the row selected) by θ_1 as illustrated in equation 11.3.

EXAMPLE 2. Suppose 10 units are available for testing (with replacement), that there are twelve days to do the test (at twenty four hours a day), and a θ_1 of 200 hours.

The total time available for testing is 2880 hours, i.e., (10 units x 12 days x 24 hours a day). Note: You may not need all these hours for testing but the test cannot exceed 2880 hours.

To find the calculated test time multiplier, divide the total time available by the lower test MTBF (θ_1), i.e.,

$$\text{Calculated test time multiplier} = 2880/200 = 14.4$$

The closest multiplier in the table for $\beta = .10$ is 14.21, and for $\alpha = .10$ the DR = 2.28, therefore, the test plan is:

$$\begin{aligned} \alpha &= .10, & \beta &= .10, \\ \theta_1 &= 200 \\ \theta_0 &= 2.28 \times 200 \text{ hours} = 456 \text{ hours and,} \\ \text{total test time} &= 14.21 \times 200 = 2842 \text{ hours} \end{aligned}$$

acceptance number = 9

Note: The equipment in this example must have an MTBF of at least 456 hours from the growth program to have at least a $(1 - \alpha)\%$, (in this case, 90%) chance of passing the test.

STOP ON A TIME OR A FAILURE: In developing a test plan decisions must be made on whether the test is stopped on a time or a failure. The Mil-HDBK-781A tables are designed for stopping on a time. However, other tables are available in which the test can be stopped on a failure. The advantage in stopping on a time is that the exact length of the test is known before the test begins which is not the case in stopping on a failure.

TEST WITH OR WITHOUT REPLACEMENT: Tests can also be run "with" or "without replacement." It is usually more efficient to run the test "with replacement" because the test hours accumulated for each calendar hour will be greater than they would be if testing were "without replacement." However, "with replacement," implies that corrective maintenance can be performed, or spare test units are available to replace failed test units.

It should also be noted that it is permissible to replace failed units with new units at any time during the test only if the failure distribution is constant as it is in the exponential.

11.14 Operating Characteristic (OC) Curve

The OC curve is a graphical illustration of the test plan. It shows the probability of passing a test (or probability of acceptance, P_a ,) for any MTBF. See Figure 1.

The OC curve is useful in illustrating the probability of passing a test for a given MTBF. This is done by entering the MTBF (horizontal) scale with an MTBF. Go straight up to the OC curve and then go directly to the P_a scale to read the probability. At θ_1 the probability should be around β ; at θ_0 the probability should be around $1 - \alpha$.

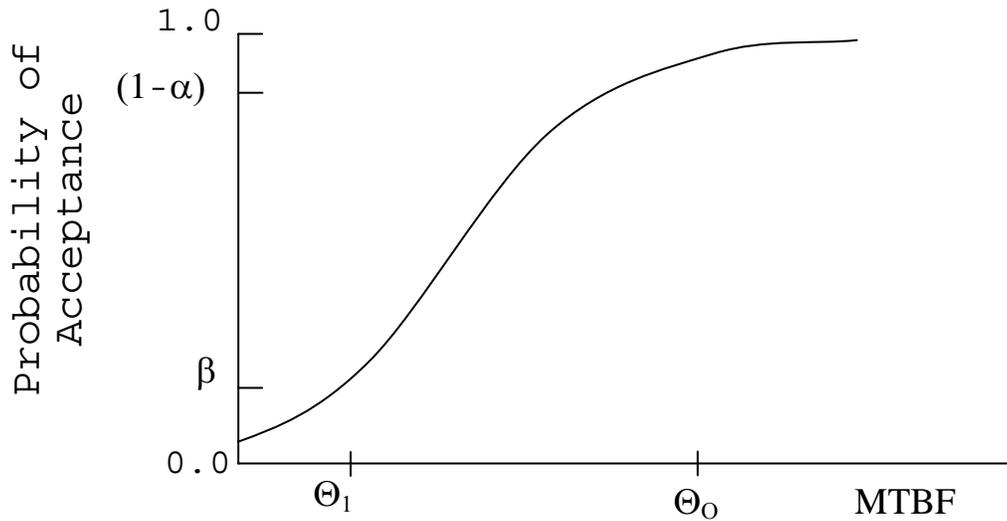


Figure 1. OC Curve for a Test Plan

Constructing An OC Curve:

Constructing an OC curve makes use of the Poisson table and the Mil-HDBK-781 test plan data. The steps are as follows:

1. Define the test plan:

$\Theta_0 = \underline{\hspace{2cm}}$, $\Theta_1 = \underline{\hspace{2cm}}$,
 Total Test Time = $\underline{\hspace{2cm}}$,
 the acceptance number = $\underline{\hspace{2cm}}$

2. Set up a table so that you end up with five points on the OC curve. The first point is Θ_1 and the last point is Θ_0 , and the three points in-between are about equally spaced between the end points.

Possible MTBF	Total Test Time	Expected # of failures	Prob. of Acceptance
Θ_1			
Θ_i			
Θ_i			
Θ_i			
Θ_0			

3. Divide the total test time by each possible MTBF to get

the expected number of failures; this is "m" for a Poisson distribution.

4. Using the "m" from step 3, and the acceptance number from test plan, find P_a from the cum. Poisson table.

5. Plot the curve, P_a vs. MTBF.

Example 3.: An $\alpha = .10$ and $\beta = .10$ test plan with a DR of 2.00 ($\Theta_1 = 200$, and $\Theta_0 = 400$) has a test time multiplier of 18.96, a total test time of 3792 (18.96 x 200) and an acceptance number of 13. The OC curve calculations are as follows:

Possible MTBF Θ_1	Total Test Time	Expected # of failures (m)	Prob. of Acceptance for c = 13
200	3792	18.96	.0998
250	3792	15.17	.35
300	3792	12.64	.61
350	3792	10.83	.80
400	3792	9.48	.899

Plotting P_a vs. MTBF results in the following graph:

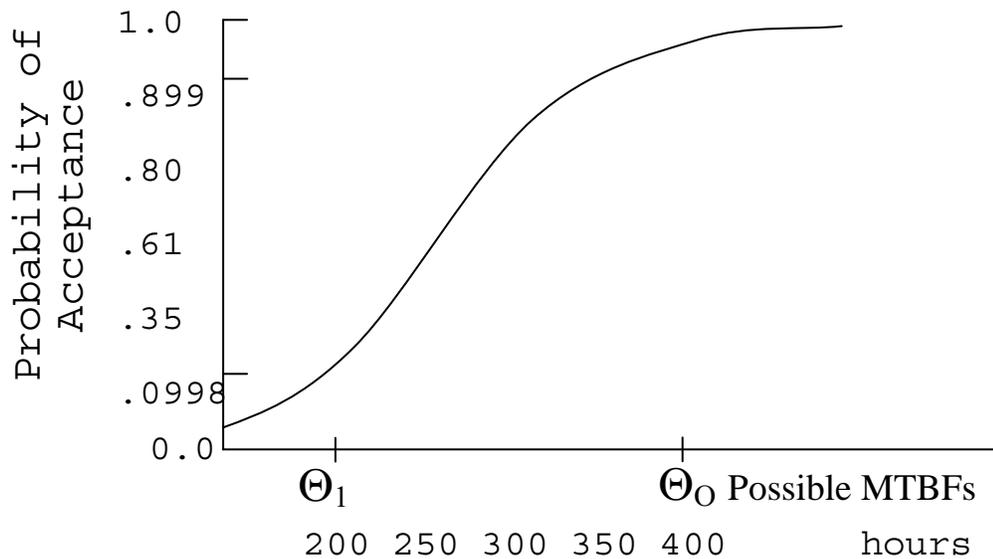


Figure 2. OC Curve for Example 3

11.15. Effects of Changing • and •

Alpha and beta are OC factors that anchor the curve on the vertical scale. Changing alpha and/or beta affects the position of the OC curve.

If alpha is increased the curve drops, this results in a decrease in the probability of acceptance for MTBFs greater than the lower test MTBF. Conversely, a decrease in alpha will cause the OC curve to rise for MTBFs greater than the lower test MTBF. The changes in the curve are related to the total test time; a smaller alpha results in a larger test time and a larger alpha results in a smaller test time.

If beta is made larger the OC curve rises for MTBFs less than the upper test MTBF, and the test time decreases; if beta is made smaller the OC curve drops for MTBFs less than the upper test MTBF and the test time increases.

For individuals involved in the selection of a test plan it is useful to know how the selection of alpha and beta affects the probability of acceptance and the total test time.

11.16. System MTBF and Upper Test MTBF

Another factor related to the derivation of a test plan is the relationship between the MTBF of the system and the upper test MTBF, θ_o . It is desirable to have a high chance of passing both RQT and PRAT tests. To achieve a high chance of passing these tests the MTBF of the system should be at least equal to θ_o . The MTBF of the system is the MTBF that results from the Reliability Development Growth Test (or TAAF). The symbol for that MTBF is M_p .

The two MTBFs, M_p and θ_o , could be exactly the same if θ_o is chosen first and the M_p is grown to that same value. However, if M_p is chosen first it is unlikely that the test plan table has a DR that is exactly the same as what was calculated. If this happens, make sure M_p is larger than θ_o so that the probability of acceptance is at least as large as $1 - \alpha$.

11.17 Relationship to Growth Testing

To find the probability of acceptance for the MTBF that was achieved by the growth test program, M_p , enter the OC curve scale with M_p and read the P_a from the probability scale.

EXAMPLE 4:

Using the data in the previous example, suppose that the M_p achieved by the growth test program is 350 hours, what is the P_a ?

Enter the MTBF scale in Figure 3 at 350 hours, go up to the curve and read the P_a on the probability scale. It should be just below .90 since 350 hours is less than 400 hours. See Figure 3 below.

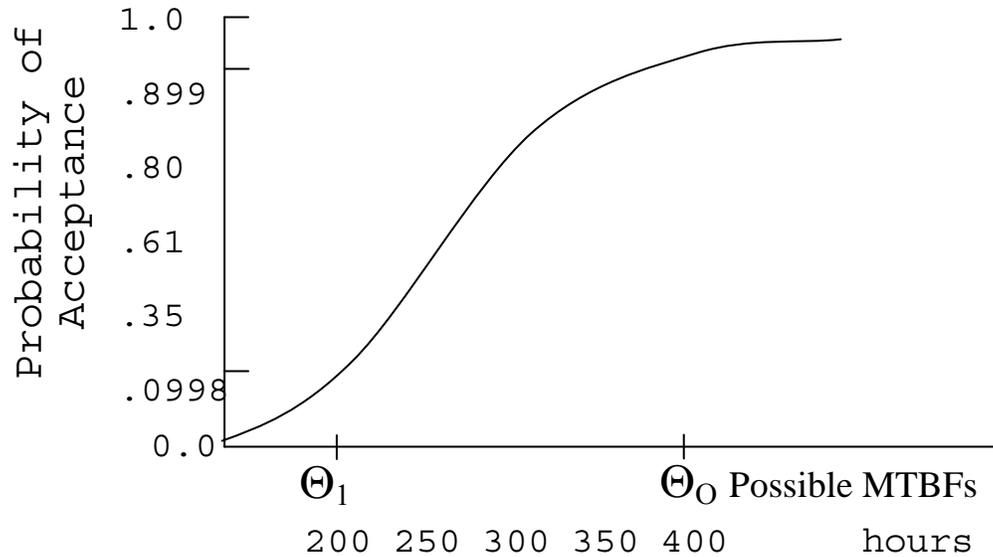


Figure 3. OC Curve for Example 4

What if M_p is 450 hours, what is P_a ?

Enter the MTBF scale at 450 and find the P_a ; the probability of acceptance should be higher than .90 since 450 is larger than 400. See Figure 3.

11.18. MTBF from Spec to PRAT

At different stages of the development and production process the MTBF has different meanings and as a result different values.

STAGE 1. SPECIFIED MTBF: The first MTBF encountered is in the equipment specification; it is a minimum MTBF that must be achieved to satisfy the customer of the product to be delivered.

STAGE 2. PREDICTED MTBF: Next comes an MTBF that is arrived at

through the prediction process. The predicted MTBF must be equal to or greater than the specified MTBF. If a confidence statement is part of the reliability specification the predicted MTBF may have to be about two times larger (or more) than the specified MTBF, otherwise it is not likely the equipment will pass the RQT.

STAGE 3. GROWTH TEST M_F : The predicted MTBF is the input to the reliability growth model. and M_F is the targeted output of the growth process.

STAGE 4. RQT TEST P_a : M_F is the MTBF that is input to the OC Curve. It is used to determine the probability of acceptance for RQT.

STAGE 5. PRAT TEST MTBF: For the PRAT test, M_F , must be adjusted to reflect the effectiveness of ESS on the system MTBF. The PRAT test M_F is calculated by dividing M_F by the ESS effectiveness factor.

11.19. Sample Size

The sample size for exponential testing can theoretically be as small as one or be the entire population, but for practical purposes the entire population would not be used for testing unless it is small.

The sample size determination should be an economic decision and be based on waiting time cost (while test is in progress), the cost of test fixtures, the time available for testing, the cost of the unit being tested (if more are built just to run the test), and other factors (e.g., restrictions on sample size).

For example, if 10 units are to be tested for the total test time of 3792 hours (Example 3) then each unit would run for 379.2 hours. This answer is found by dividing the total test time by the number of units on test:

$$\text{Test time per unit} = \text{total test time}/\text{sample size} \quad (11.4)$$

$$\text{Test time per unit} = 3792 \text{ hours}/10 = 379.2 \text{ hours}$$

If the sample size were 100, then

$$\text{Test time per unit} = 3792/100 = 37.92 \text{ hours.}$$

11.20. Selecting the Sample

In selecting the sample it is important that each unit produced have an equal chance of being selected. This can be accomplished by selecting the sample from a completed lot, or assigning random numbers to each production unit and then

selecting numbers from a random number table to determine which production units to select.

Sampling could also be done by dividing the lot into equal subgroups and then selecting the same number of units from each subgroup. For example, if 100 units are produced each month and a sample of 20 is desired, the 100 units could be divided into four groups of 25, then 5 could be taken at random from each of these subgroups to get the 20 sample units.

In conducting the test, it is not critical that the number of hours be exactly the same on each test unit. On the other hand it is not wise to run a few hours on several units and a large number of hours on a few units. In Mil-HDBK-781 there is a requirement that the smallest amount of time on any one unit cannot be less than half the average of the time on all the units. The purpose of this restriction is to prevent someone from putting several units on for a short time just to pass the test.

11.21. ESS

In testing of electronic products it is important that the products be screened before starting the test. The screen is called ESS, Electronic Stress Screening. This procedure is discussed in detail in a later chapter.

11.22. Relationship between Testing and Confidence

The "Total Test time (Multiples of θ_1)" from the table shown earlier is half of the Chi-square value for $2(c + 1)$ degrees of freedom on the Chi-square table. For example, for $\beta = .10$ and $c = 13$ the multiplier is 18.96; the Chi-square value for 28 degrees of freedom and 90% confidence is 37.9 which when divided by two is 18.95.

Another point of interest is that the lower confidence limit, for a test in which the number of failures that occurs is the same as the acceptance number, is the lower test MTBF.

For example, suppose that the test plan for a system is as follows:

$$\theta_0 = 300 \text{ and } \alpha = .05$$

$$\theta_1 = 100 \text{ and } \beta = .10$$

$$DR = 300/100 = 3.0$$

The test time multiplier is 11.77 and the acceptance number is 7. Total test time is:

$$11.77 \times 100 \text{ hours} = 1177 \text{ hours}$$

If the test is run what would the lower confidence limit be for θ if 7 failures occurred?

The Chi-square value for 16 degrees of freedom (df) and 90% confidence is 23.5. Note: $df = 2(c + 1)$ and $c = 7$. Using the Chi-square model,

$$\text{lower confidence limit} = 2(1177)/23.5 = 100+$$

which is θ_1 . This shows that a test plan could be developed from the Chi-square table by dividing the Chi-square value by 2 and multiplying by the lower test MTBF. For example, divide 23.5 by 2 and you get 11.75; now multiply by 100 to get 1175 hours which is basically what we started with; the acceptance number is the degrees of freedom minus 2 and divided by 2. That is, 23.5 has 16 degrees of freedom, so 7 [i.e., $(16 - 2)/2 = 7$] is the acceptance number; it is the same as what we started with earlier.

Table: Chi-Square Critical Values

The areas given across the top are the areas to the right of the critical value.

df	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	---	---	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582

20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

11.23 Binomial Testing

For discrete measurements binomial tests apply. Since the calculations for binomial probabilities, confidence limits and sample sizes for reliability testing have already been presented in Section Four, they will not be repeated in this section.

BINOMIAL NOMOGRAPH the Nomograph of the Cumulative Binomial Distribution can be used to find a test plan. The procedure is as follows:

1. Define $1 - \beta$, and the specified Reliability.
2. Define α and the upper test Reliability.
3. On the Binomial Cumulative Nomograph locate the confidence level scale and the $1 - \beta$ point on this scale.
4. On the opposite side of the nomograph locate the reliability scale and the specified reliability on this scale.
5. Draw a straight line between the points on these scales, and call this line #1.

6. On the Binomial Cumulative Nomograph locate the confidence scale and the α point on this scale.

7. On the opposite side of the nomograph locate the reliability scale and the upper test reliability on this scale.

8. Draw a straight line between the points on these scales, and call this line #2.

9. Where line #1 intersects line #2 read the number of failures and the sample size.

EXAMPLE 5 A system has a test requirement that a 95% reliability be demonstrated with 90% confidence; and an upper test reliability of .99 for which $\alpha = .10$. What Binomial test plan could be used?

When line #1 is drawn to connect the two points, 90% confidence and 95% reliability, the line crosses several test plans. For example, at:

zero failures, about 45 trials would be required,
one failure, about 78 trials would be required,
two failures, about 107 trials would be required, etc.

Any of these test plans could be chosen and would meet the confidence and reliability requirement if only β and the lower test reliability were given. Where the test plans differ is at the other end of the OC curve, the $[P_o, (1-\alpha)]$ coordinates. For example, at:

Zero failures, $\alpha = .1$ and $n = 45$, $P_o = .0022$ because:
 $1 - \alpha = .9 = P(0) = {}_n C_0 p^0 (1-p)^{n-0} = (1)(1)(1-p)^n = (1-p)^{45} \Rightarrow p = .0022$

One failure, $\alpha = .1$ and $n = 78$, $P_o = .0068$ because:
 $1 - \alpha = .9 = P(0 \text{ or } 1) = P(0) + P(1) = (1-p)^n + np^1(1-p)^{n-1}; n = 78 \Rightarrow p = .0068$

Note: $P(0) + P(1) = .5873 + .3136 = .9009 \cong .90$

Two failures, $\alpha = .1$ and $n = 107$, $P_o = .0102$.

As you can see, a small number of trials results in a smaller P_o for a given α . Therefore, if P_o and α are not specified, any one of these test plans would be acceptable, but if P_o and α are specified then that information must be used in designing a test plan.

In this example, α is defined as .10 for an upper test reliability of .99. When the line #2 is drawn it crosses line #1 at $r = 2$ (failures) and at about 110 trials (n). Hence, it appears that the test plan is to conduct 110 trials; if two or less failures occur, the equipment has passed the test. See

Figure 4.

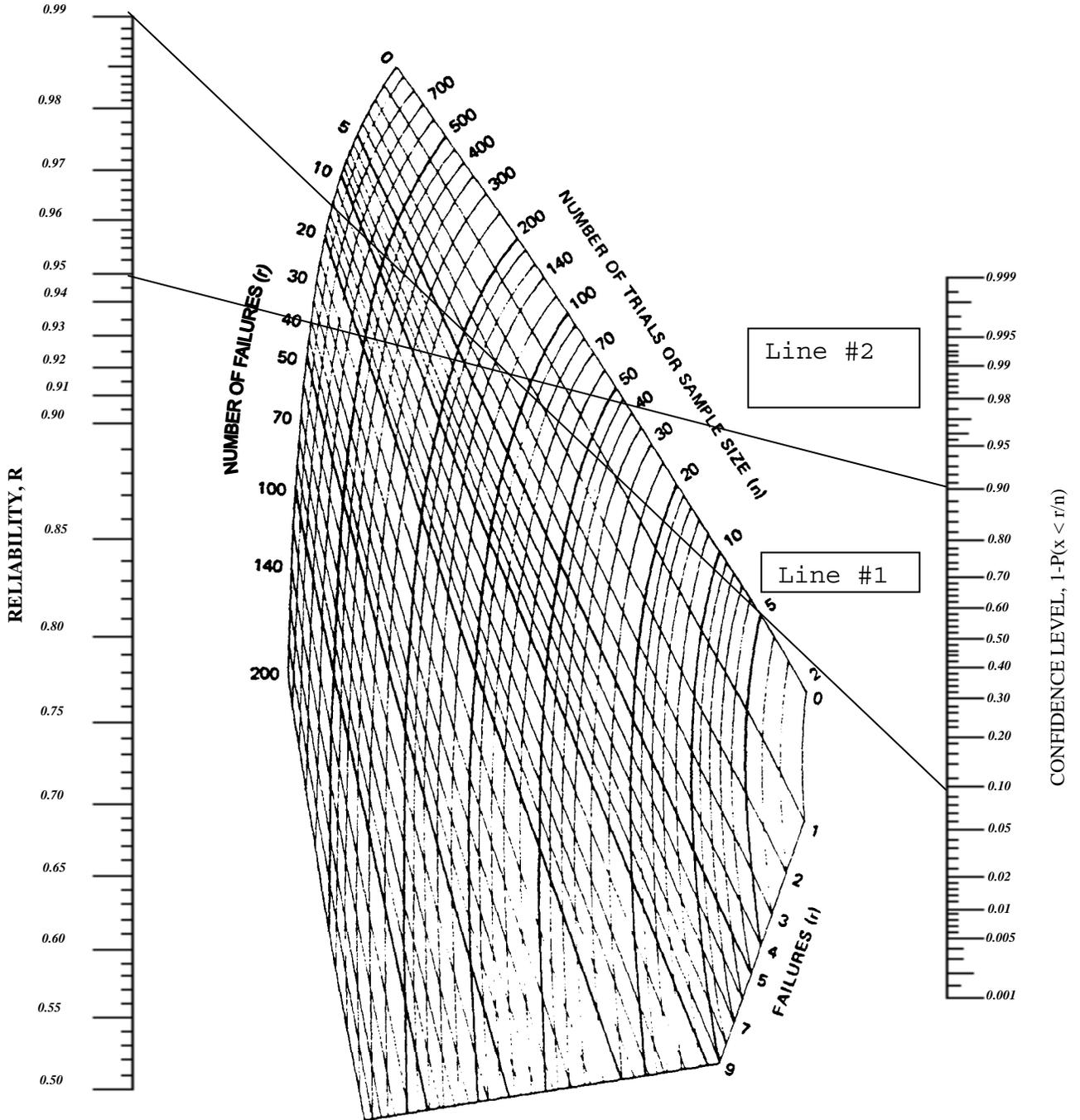


Figure A-6. Nomograph of the Cumulative Binomial Distribution

Adapted from Figure 1, "Graphical Determination of Single-Sample Attribute Plans for Individual Small Lots," by Shaul P. Ladany,

Figure 4 Using the Nomograph to Find a Test Plan.

TABLE METHOD: To find a Binomial test plan using the table for Fixed Length Test Plans for the Exponential Distribution the following procedure can be used:

1. Define α , β , P_0 and P_1 .

α and β are as defined earlier,
 $P_1 = 1 - R_1 = 1 -$ specified reliability, (this is the reliability that corresponds to β), and
 $P_0 = 1 - R_0 = 1 -$ the reliability that corresponds to $1 - \alpha$.

2. Compute the discrimination ratio (DR) as P_1/P_0 (not as accurate as $DR = \frac{\ln R_1}{\ln R_0}$).

3. Using the Exponential test table, find the α and β column; go down that column until you come to a DR that is equal to or less than the DR computed in step 2. If the DR in the table is less than what was computed, the table DR becomes the new DR and will be used in a later step.

4. In the same row of the new DR, write down the acceptance number (c) and the nP_1 value (this will be found in Column 2 "Total Test time (Multiples of θ_1)").

5. Divide nP_1 by P_1 , this gives you the number of trials (n) to use for the test.

6. Find the new P_0 , by dividing P_1 by the new DR.
Note: A new P_0 is required because it is not possible to get the exact DR desired for the original P_0 .

7. Test plan is:

Conduct n trials
If c or fewer failures occur, the test has been passed.

Check: To check the plan, compute $(n \times \text{new } P_0)$, and find this value in the "m" or "np" column of the Poisson table. Find the probability of acceptance in the "c" column. The probability should be about $(1 - \alpha)$.

Now multiply $(n \times P_1)$ and find this value in the "m" column of the Cumulative Poisson table. Find the probability of acceptance in the "c" column. The probability should be close to

β .

EXAMPLE 6. A Binomial test plan is desired to demonstrate a reliability of .95 with 90% confidence; the chance of accepting a reliability of .99 is also set at 90%.

1. $\alpha = .10$ and $\beta = .10$. Note: β always goes with the smaller reliability, in this case .95 reliability; and α goes with the .99 reliability.

$$\begin{aligned} P_0 &= 1.00 - .99 = .01 \\ P_1 &= 1.00 - .95 = .05 \end{aligned}$$

2. $DR = .05/.01 = 5$

3. From the Exponential test table for $\alpha = \beta = .10$, the first DR that is equal to or less than 5, is 4.83. This is the new DR.

4. $nP_1 = 5.32$ (Column 2), and $c = 2$.

At this point, we have:

$$\begin{aligned} \text{new DR} &= 4.83 \\ c &= 2 \\ nP_1 &= 5.32 \end{aligned}$$

5. The number of trials, n , is found as follows:

$$n = nP_1/P_1 = 5.32/.05 = 106.4 = 107$$

6. New $P_0 = P_1/\text{new DR} = .05/4.83 = .0104$

7. The test plan is: conduct 107 trials, if 2 or less failures occur the test has been passed.

Check:

$$\begin{aligned} nP_0 &= (107)(.0104) = 1.107; \text{ from the Poisson for } c=2, P_a = .90. \\ nP_1 &= (107)(.05) = 5.35; \text{ from Poisson for } c = 2, P_a = .0985. \end{aligned}$$

Note: P_a is less than .10 because "n" was rounded up to 107; if 106.4 could be used as "n" the P_a would be .1006.

CONSTRUCTING AN OC CURVE FOR A BINOMIAL TEST:

Constructing an OC curve makes use of the Poisson table, the number of trials, the acceptance number, and an upper and lower test unreliability. The steps are as follows:

1. Define the test plan:

Lower test unreliability (P_1) = _____

Upper test unreliability (new P_0) = _____

Number of trials = _____

Acceptance number = _____

2. Set up a table so that you end up with five points on the OC curve. The first point is the lower test reliability P_1 and the last point is the upper test reliability P_0 ; the three points in-between are about equally spaced between the end points.

Possible Unreliab.	Number of Trials	Expected # of failures	Prob. of Acceptance
P_1			
P_i			
P_i			
P_i			
P_0			

3. Multiply the unreliability by the number of trials to get the expected number of failures, this is "m" for a Poisson distribution.

4. Using the "m" from step 3, and the acceptance number from the test plan, find the P_a from the Poisson table.

5. Plot the curve, P_a vs. Reliability.

EXAMPLE 7. It was desired to have a binomial test plan in which $\alpha=\beta=.10$ and in which $R_1=.95$ and $R_0 = .99$. Applying the Cumulative Binomial Nomograph, the following binomial test plan was derived: allow 2 failures in 107 trials. Construct the corresponding OC curve using the Poisson distribution.

$P_1=1-R_1=1-.95=.05$ and $P_0=1-R_0=1-.99=.01$ Using the same format presented above, we find:

Possible Unreliab.	Number of Trials	Expected # of failures	Prob. of Acceptance
.05	107	5.35	.0985
.04	107	4.28	.20
.03	107	3.21	.38
.02	107	2.14	.66
.01	107	1.11	.90

The approximate OC curve for this test plan is illustrated

below. Note: When applying the Poisson distribution to construct an OC curve for a binomial test plan, it is usually not possible to find the exact P_a desired, e.g. we found .0985, rather than .10.

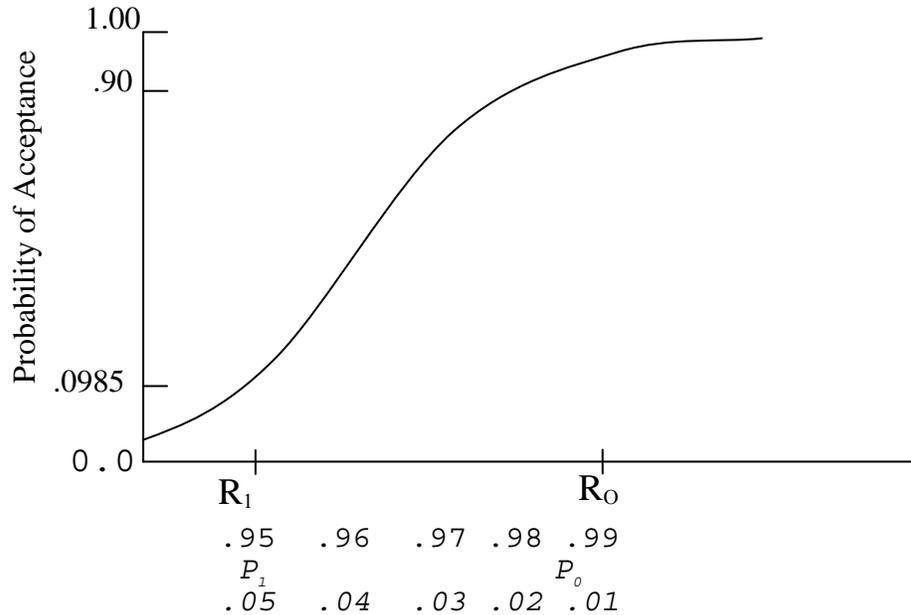


Figure 5. OC Curve for a binomial test plan based on Poisson Distribution calculations.

11.24 USING EXPONENTIAL TEST PLAN TABLES TO DEVELOP BINOMIAL TEST PLANS

PROBLEM:

- Cumulative binomial nomograph limited to $R \leq 0.99$
- Cumulative binomial tables not readily available; require iterative solution

➤ SOLUTION:

- Both Exponential and Binomial distributions assume **constant** failure rate or **constant** probability of success/failure.

- Can use fixed-length, exponential test plan process to determine binomial test plans

THEORY

- Fixed-length exponential test plans based on $DR = \Theta_0/\Theta_1$
- Need method to relate $P(\text{success})$ to Θ s and DR

$$P(\text{success}) = R(t) = e^{-t/\Theta}$$

$$\ln [R(t)] = -t/\Theta \quad (11.5)$$

$$\Theta = -t/\ln [R(t)] \quad (11.6)$$

Therefore: $\Theta_0 = -t/\ln [R_0(t)]$, and

$$\Theta_1 = -t/\ln [R_1(t)]$$

where R_0 and R_1 are the desired and minimum acceptable values. Substituting:

$$DR = \frac{-t/\ln [R_0]}{-t/\ln [R_1]} = \frac{\ln [R_1]}{\ln [R_0]} \quad (11.7)$$

EXAMPLE 8

- Minimum reliability requirement = $R_1 = 0.85$
- Desired reliability = $R_0 = 0.92$
- β risk = 10%
- α risk = 20%
- Question: What is an appropriate test plan - number of trials (n) and maximum number of mission failures (C)?

$$DR = \frac{\ln [R_1]}{\ln [R_0]} = \frac{\ln 0.85}{\ln 0.92} = \frac{-0.1625}{-0.08338} = 1.95$$

From the (exponential) fixed-length test plan table given earlier in this chapter:

- Max number of failures = 9
- multiplier = 14.21

$$\text{Total test time} = 14.21 \times \Theta_1$$

$$\text{Note that number of trials/tests, } n = \frac{\text{Total test time}}{\text{Mission length (t)}}$$

$$\text{Therefore, } n = \frac{14.21 \times \Theta_1}{t}$$

But, $\Theta_1 = -t/\ln[R_1]$, therefore:

$$n = \frac{14.21\{-t/\ln[R_1]\}}{t} = \frac{-14.21}{\ln[R_1]}$$

$$= \frac{-14.21}{\ln 0.85} = \frac{-14.21}{-0.1625} = 87.4 \text{ trials}$$

➤ As a check, a similar answer can be derived from the nomograph.

EXAMPLE 2

- Minimum requirement = $R_1 = 0.995$ (not on nomograph)
- Desired reliability = $R_0(t) = 0.997$
- β risk = 10%
- α risk = 20%
- Question: What is an appropriate test plan?

$$DR = \frac{\ln [R_1]}{\ln [R_0]} = \frac{\ln 0.995}{\ln 0.997} = \frac{-0.00501}{-0.003005} = 1.67$$

Appropriate exponential test plan:

- Max number of failures = 16
- Test time multiplier = 22.45

$$n = \frac{-22.45}{\ln [R_1]} = \frac{-22.45}{-0.00501} = 4479 \text{ trials}$$

➤ In this case, the answer can NOT be derived from the nomograph.

11.25 Summary

This chapter explained several concepts that are part of a testing program. Some aspects of a test program are not discussed in detail because emphasis is placed on the development of the RQT and/or PRAT test plans for both exponential and binomial data.

From this section the reader should know what is involved in developing an exponential test plan, a binomial test plan and know what decisions need to be made in setting up a Quality Assurance test plan for production units. For the QA test plan the form on the following page may be helpful.

11.26. Equations

$$DR = \theta_0 / \theta_1. \quad (11.1)$$

$$MTBF = -(\text{mission time}) / \ln(\text{Reliability}) \quad (11.2)$$

$$\theta_0 = \theta_1 \times DR \quad (11.3)$$

$$\text{Test time per unit} = \text{total test time} / \text{sample size} \quad (11.4)$$

$$\ln [R(t)] = -t/\theta \quad (11.5)$$

$$\theta = -t / \ln [R(t)] \quad (11.6)$$

$$DR = \frac{-t / \ln [R_0]}{-t / \ln [R_1]} = \frac{\ln [R_1]}{\ln [R_0]} \quad (11.7)$$

Chapter 12 (Omitted)

Chapter 13
Normal Distribution
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Symbols, Terms And Equations Used In This Section:

- μ = population mean for a Normal distribution
- σ = population standard deviation for a Normal distribution
- LPL = lower process limit
- UPL = upper process limit
- PL = process limits
- USL = Upper specification limit
- LSL = Lower specification limit
- Z = Standard normal deviate
- x = observation from a population or sample

standard deviation = a measure of variation
 standard error of the mean = a measure of the variation for a distribution of averages
 population = the entire output of a process for a given set of parameters
 sample = a portion or subset of a population

13.1. Introduction

The Normal distribution is a useful distribution because it has many applications in manufacturing and administrative processes. It is also used to approximate other distributions under certain conditions.

In this chapter the emphasis will be on the following:

1. Properties of the Normal.
2. Calculating the mean and standard deviation.
3. Areas under the Normal curve.
4. Distribution of sample means.
5. Estimating sample sizes.

13.2. Properties of the Normal Distribution

The Normal distribution is a continuous distribution and has the following properties (see Figure 13.1):

1. It is defined by its mean, μ , and standard deviation, σ .
2. It is symmetrical around the mean.
3. The area from minus infinity to plus infinity is one.
4. It has an increasing failure rate.
5. Areas around the mean are:

$\mu \pm 1\sigma$ include 68.26% of the total area,
 $\mu \pm 2\sigma$ include 95.44% of the total area,
 $\mu \pm 3\sigma$ include 99.72% of the total area.

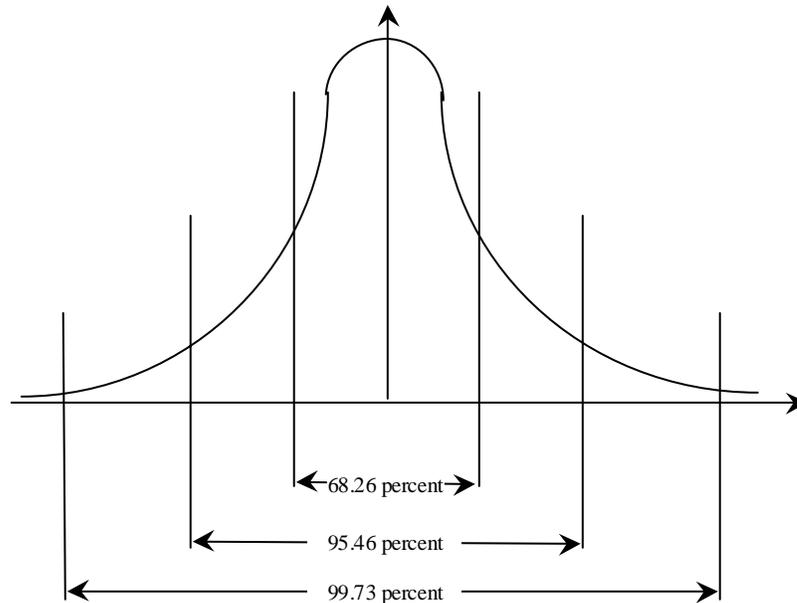


Figure 13.1 The Normal Curve.

13.3. Computations

POPULATION MEAN: The mean is one of three measures of central tendency for a distribution, such as the Normal. The others are: the median which is the point that divides a Normal distribution exactly in half; and the mode which is the observation with the largest frequency. In this course our main concern will be with the mean.

To compute the mean for the population the following equation is used:

$$\mu = \frac{\sum x_i}{N} \quad (13.1)$$

where,

μ = the mean of the population

x_i = the value of the "ith" observation

$\sum x_i$ = the sum of all the observations

N = size of the population

POPULATION STANDARD DEVIATION: The standard deviation is a measure of variation for the Normal distribution. Other measures of variation include the variance, which is the square of the standard deviation; and the range. The range will be used in the chapter on Statistical Process Control.

Variation is a measure of how close the observations are clustered around the mean. When the variation is large the observations are spread out over a relatively wide range around the mean; when the variation is small the observations are more closely bunched around the mean.

A small amount of variation is more desirable than a large amount for several reasons. When the variation is small, you get: better predictions, better process capability numbers, lower cost on the Taguchi Loss Function, less sampling error, better accuracy. In reliability, small variation is also desirable because it results in higher reliability and longer life for a system.

The standard deviation is computed by:

- 1- finding the deviation of each observation from the mean,
- 2- squaring the deviation, then
- 3- summing the squared deviations,
- 4- dividing by the population size, N , and then
- 5- taking the square root of the result.

In equation form, it looks like this:

$$\sigma = [\sum(x_i - \mu)^2/N]^{1/2} \quad (13.2)$$

where,

$x_i - \mu$ = the deviation of the "ith" observation from the mean,

$(x_i - \mu)^2$ = the square of a deviation,

$\Sigma(x_i - \mu)^2$ = the sum of all the squared deviations,

N = the number of observations in the population, or population size.

EXAMPLE 13.1: Consider a population of 4 members with the values: 5,8,2, and 9.

These four observations have a mean of 6; i.e., $5 + 8 + 2 + 9 = 24$, which, when divided by 4, results in a mean of 6, i.e., $\mu = 6$

Suppose that now we subtract the mean from each observation and then square those differences. we get:

x_i	$x_i - \mu$	$(x_i - \mu)^2$
5	$5 - 6 = -1$	$(-1)^2 = 1$
8	$8 - 6 = 2$	$(2)^2 = 4$
2	$2 - 6 = -4$	$(-4)^2 = 16$
9	$9 - 6 = 3$	$(3)^2 = 9$
$\Sigma x_i = 24$	$\Sigma(x_i - \mu) = 0$	$\Sigma(x_i - \mu)^2 = 30$

and a standard deviation of:

$$\sigma = \sqrt{\frac{30}{4}} = \sqrt{7.5} = 2.7$$

It is important to note that the sum of the deviations around the mean is zero, i.e., $\Sigma(x_i - \mu) = 0$, because this is a property of the mean. There is only one number for which the sum of the deviations around it is zero, and that number is the mean.

13.4. Sample Statistics

SAMPLE MEAN: It is very unusual to be able to work with an entire population. In most situations the only data available is a sample drawn from the population. Consequently, the sample mean and sample standard deviation are used as estimates of the population mean and standard deviation. The sample mean and sample standard deviation are called statistics.

The equation for the sample mean is as follows:

$$\bar{x} = \frac{\sum x_i}{n} \quad (13.3)$$

where,

\bar{x} = the mean of the sample

n = the sample size,

x_i = the "ith" observation in the sample, and

$\sum x_i$ = the sum of the sample observations.

SAMPLE STANDARD DEVIATION: The sample standard deviation looks similar to the population standard deviation with two exceptions: n, the sample size, is used in place of the population size, N; and the divisor, which is called the degrees of freedom, is "n-1." Every time one statistic is used to calculate another statistic a degree of freedom is lost. The "1" that is subtracted from "n" represents the one degree of freedom that is lost when the sample mean is used to calculate a sample standard deviation.

The equation for the sample standard deviation has been algebraically simplified to the following:

$$s = \sqrt{\frac{[\sum x^2] - \frac{[\sum x]^2}{n}}{n-1}} \quad (13.4)$$

EXAMPLE 13.2: A sample of five parts selected from a population has the following dimensions: 10, 8, 11, 9, 10

$$\bar{X} = [\sum x]/5 = (10 + 8 + 11 + 9 + 10)/5 = 48/5 = 9.6$$

To find s using equation 13.4 we begin by computing:

$$\sum(x_i^2) = 10^2 + 8^2 + 11^2 + 9^2 + 10^2 = 100 + 64 + 121 + 81 + 100 = 466$$

$$s = \sqrt{\frac{466 - \frac{[48]^2}{5}}{5-1}} = \sqrt{\frac{466 - \frac{2304}{5}}{4}} = \sqrt{\frac{466 - 460.8}{4}} = \sqrt{\frac{5.2}{4}} = \sqrt{1.3} = 1.14$$

\bar{X} and s are estimates μ and σ . Are they good estimates? That depends on how you use the estimate, the sample size, n , and how the samples are drawn from the population. A sample size of 100 drawn in an unbiased manner should result in a good estimate.

13.5 Area Under the Normal Curve

One of the calculations that we will be making is to compute areas under the Normal curve for quality and reliability problems. The model used is derived from the density function which is defined as follows:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad (13.5)$$

This expression could be used to calculate areas under the Normal curve but the Normal Table has simplified the process. The Table makes use of the portion of the density function that contains the variable "x," every other term is a constant. The portion containing "x" has been defined as "Z," and is called the standard normal deviate:

$$Z = (x - \mu)/\sigma \quad (13.6) \text{ and}$$

$$Z = (x - \bar{x})/s \quad (13.7)$$

By making this transformation from "x" to "Z" the scale of the Normal distribution is being changed from "x" to "Z" which can be used for any type of measurement. See Figure 13.2

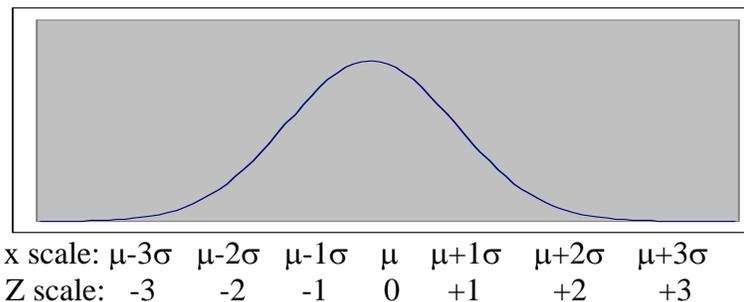


Figure 13.2 Normal Distribution with x-scale and Z-scale.

From this scale it should be apparent that "Z" represents the number of standard deviations that an observation is from the mean, so that when Z is calculated the answer

is in "standard deviations from the mean." The Normal table in the Appendix provides areas from minus infinity to +Z.

To use the Normal table, select a value for which you want to know the probability of being above or below that value. For example, I might ask, "What is the probability that a system will wear out before 1000 hours of use?" The "1000 hours" point is converted to "Z" using equation 13.7, and the probability of wear out before 1000 hours is found in the Normal table. Our table provides areas from minus infinity to the value of interest.

A portion of the table is presented:

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517						
1.0	.8413	.8438	.8461							

To find Z in the table, the row and column headings are used beginning with the row heading. For example,

Z = 0.05 is in the 0.0 row and the .05 column, and the value, .5199 is the area from minus infinity to Z = .05;

Z = .11 is in the 0.1 row and the .01 column, and the value .5438 is the area from minus infinity to z = .11. It is possible to find areas above or below Z simply by subtracting the area below Z from 1.0000. For example, if the area to the right of Z = .11 is what you want, just subtract .5438 from 1.0000, i.e., $1.0000 - .5438 = .4562$.

Z = 1.02 is in the 1.0 row and the .02 column, and the value .8461 is the area from minus infinity to Z = 1.02. The area to the right of Z = 1.02 is .1539.

EXAMPLE 13.3: A process has a mean of 104 and a standard deviation of 4.

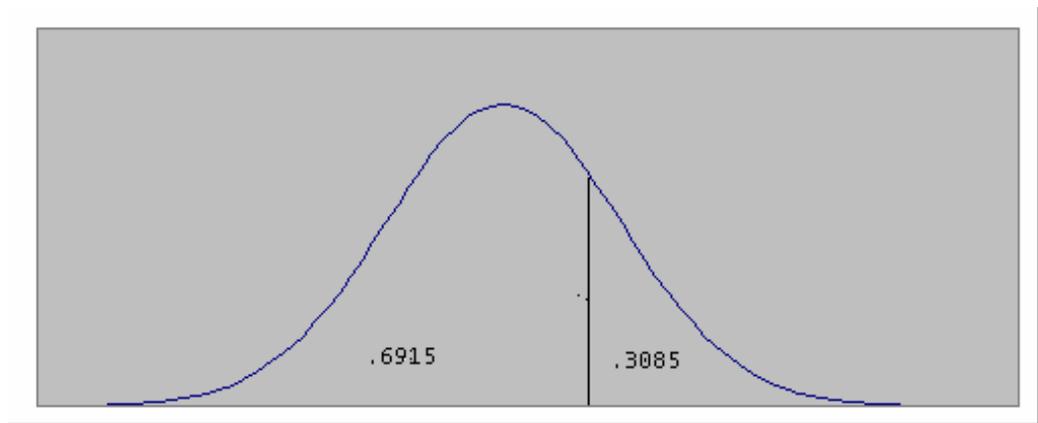
a. What per cent of the process is below 106?

$$Z = \frac{106 - 104}{4} = 2/4 = .50$$

From the Z table the area is .6915, so 69.15% of the process is less than 106.

b. What per cent of the process is above 106?

The area below 106 is .6915, thus, the area above 106 is .3085, which is found by subtracting .6915 from 1.0000. See Figure 13.3



					106		
x	92	96	100	104	108	112	116
Z	-3	-2	-1	0	+1	+2	+3

Figure 13.3 Area Under a Normal Curve for Example 13.3

EXAMPLE 13.5: For the same data in the previous example, a mean of 104 and a standard deviation of 4, what is the probability that an observation drawn at random will be greater than 97?

This statement can be written as:

$$P(x > 97) = ?$$

Using the same model as above,

$$Z = \frac{97 - 104}{4} = -7/4 = -1.75$$

Since Z is negative we know Z is in the left tail; the area could be on either side, but the question in the problem is asking for the area above 97.

Now, if Z were positive, i.e., if $Z = + 1.75$, the table value would provide us with the area below $Z = 1.75$; since Z is negative, the table is providing us with the area above -1.75 or above 97 and that area is .9599. This interpretation can be made because the curve is symmetrical around the mean. The area below a positive Z is the same as the area above the negative Z.

If the area to the left of 97 were desired, the area would be found by subtracting .9599 from 1.0000 or .0401.

In general the following rules can be established:

Z	Area Desired	Rule
+	Below Z	The area in the table is the answer.
+	Above Z	Subtract the area in table from 1.000.
-	Below Z	Subtract the area in table from 1.000.
-	Above Z	The area in the table is the answer.

SOLVING FOR X: In some situations the area of the curve, or probability, is known and the value of "x" must be found. In that case, the Z equation is solved for "x" instead of "Z." Solving the Equation 13.7 for "x" results in the following model:

$$x = (Z)(s) + \bar{X} \quad (13.8)$$

13.6 Process Limits

A term that will often be used is "process limits;" they correspond to plus and minus three standard deviations, that is,

$$\text{UPL} = \text{Upper process limit} = \text{Mean} + 3 \text{ standard deviations}$$

$$\text{LPL} = \text{Lower process limit} = \text{Mean} - 3 \text{ standard deviations}$$

and they can be written as:

$$\text{Process Limits} = \text{Mean} \pm 3 \text{ standard deviations} \quad (13.9)$$

13.7 Specification Limits

Specification (spec) limits will also be used in some applications. These limits that are defined by the customer, engineering, or other functions within an organization. The specs could come from an engineering drawing, a contract, a requirement, etc. and are identified by the symbols USL and LSL, where,

$$\text{USL} = \text{Upper specification level}$$

$$\text{LSL} = \text{Lower specification level}$$

In the ideal situation the process limits should fall inside the specification limits.

13.8. Distribution of Sample Means

Another Normal distribution that has application in reliability and quality is the distribution of sample means. This distribution will contain some differences from the population distribution that was discussed in the first part of this section because it is a distribution of sample means.

The population distribution consists of all the individual observations, and the distribution of sample means consists of all the sample means that are possible from the population.

STANDARD ERROR OF THE MEAN: The distribution of sample means has the same average as the population but the standard deviation is different. The standard deviation of sample averages is called the standard error of the mean.

$$\text{Standard Error of the Mean} = \frac{\sigma}{\sqrt{n}} \quad (13.10)$$

EXAMPLE 13.5 To illustrate the relationship between the two distributions, consider a population with a mean of 100 and a standard deviation of 4 and how the means of samples of size 25 would be distributed.

The mean of population distribution is 100,
The standard deviation of the population is 4,
The process limits are $100 \pm 3(4) = 88$ to 112,

The mean of the distribution of sample means is 100,
The standard error of the mean is $4/\sqrt{25} = .8$
The limits on the distribution of sample means are
 $100 \pm (3)(.8) = 97.6$ to 102.4.

See Figure 13.4 for an illustration of this example:

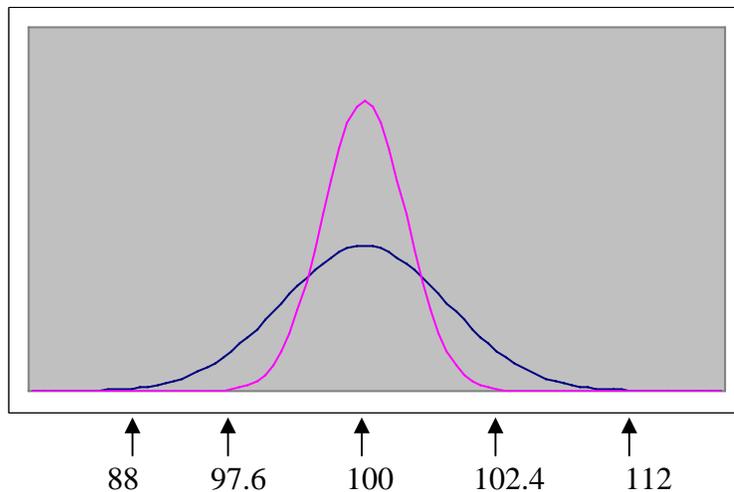


Figure 13.4 Illustration of Example 13.4

The limits for the distribution of sample averages represent the range within which 99.72% of the sample averages should fall if the population does not change. Or, to say it in another way, we would expect 99.72% of all the sample averages from samples of 25 to be between 97.6 and 102.4; or be within 2.4 of the true mean.

We have just shown how to go from a population to a distribution of sample means. Can we go in the reverse direction? That is, if we know how the sample averages are distributed can we estimate the population mean? The answer is “yes” and all we need to know is the accuracy and the confidence desired, and an estimate of the standard deviation of the population.

13.9 Estimating Sample Size

The distribution of sample averages can be used to estimate sample sizes required for testing purposes. The factors needed to make the estimate are:

1. The accuracy desired (same units as x)
2. An estimate of the population standard deviation, (this must come from past history, or from a sample drawn for the purpose of making such an estimate)
3. Confidence level (commonly used values are 90% and 95%)

The method used starts with the following relationship:

$$\text{Accuracy} = (Z_{\text{conf}}) \times (\text{Standard Error}) \quad (13.11)$$

where,

$$\text{Standard Error} = \text{Standard Deviation} / \sqrt{n}$$

where Z is found in the Normal table for the appropriate multiplier from the Normal table,

$Z_{\text{conf}} = 3$ corresponds to 99.72% confidence

$Z_{\text{conf}} = 2$ corresponds to 95.44% confidence

$Z_{\text{conf}} = 1.96$ corresponds to 95% confidence

$Z_{\text{conf}} = 1.645$ corresponds to 90% confidence

Example: For a confidence of 95%, look for .9750; it should be in the .06 column and the 1.90 row, therefore, $Z = 1.96$.

Note: .9750 is .5000 plus half of .9500.

Let us suppose that we want to take a sample from a process, and want to have 90% confidence that the sample mean is within 2 hours of the population mean. Let us also suppose that we know from past history that the population standard deviation is about 5 hours. Summarizing these facts, we have:

standard deviation = 5 hours

accuracy = 2 hours

confidence = 90% ($Z = 1.645$)

Using equation 13.11,

$$2 \text{ hours} = (1.645) \times (5 \text{ hours}) / \sqrt{n}$$

$$\sqrt{n} = (.8225 \times 5) / 2 = 4.1125$$

$$n = 16.9 \cong 17.$$

13.10. Summary

This has been a short treatise of the Normal distribution but should be sufficient for the applications in this course.

13.11. Equations

$\mu \pm 1\sigma$ includes 68.26% of the total area,
 $\mu \pm 2\sigma$ includes 95.44% of the total area,
 $\mu \pm 3\sigma$ includes 99.72% of the total area.

$$\mu = \frac{\sum x_i}{N} \quad (13.1)$$

$$\sigma = [\sum(x_i - \mu)^2/N]^{1/2} \quad (13.2)$$

$$\bar{X} = \sum x_i/n \quad (13.3)$$

$$s = \sqrt{\frac{[\sum x^2] - \frac{[\sum x]^2}{n}}{n-1}} \quad (13.4)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad (13.5)$$

$$Z = (x - \mu)/\sigma \quad (13.6)$$

$$Z = (x - \bar{x})/s \quad (13.7)$$

$$x = (Z)(s) + \bar{X} \quad (13.8)$$

$$\text{Process Limits} = \text{Mean} \pm 3 \text{ standard deviations} \quad (13.9)$$

$$\text{Standard Error of the Mean} = \frac{\sigma}{\sqrt{n}} \quad (13.10)$$

$$\text{Accuracy} = (Z_{\text{conf}}) \times (\text{Standard Error}) \quad (13.11)$$

